**Handout for the CES-Seminar Talk**

# Vortex Methods

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## 1 Motivation

Conventional methods for solving the Navier-Stokes equations, which govern fluid motion, typically require a computational mesh of the fluid domain. If the geometry of the fluid domain takes complicated shapes, mesh generation often becomes the major bottleneck in the overall simulation process. Furthermore, exterior flow problems such as the flow around an air foil, require artificial boundary conditions or other trickery.

Vortex methods are mesh-free, particle based discretisations which do not suffer from these disadvantages. *Mesh-free* in this context means, that only the boundaries of the computational domain—not the domain itself—needs to be described using a surface mesh. This greatly simplifies the meshing process and allows infinite domains.

The aim of this text is to give an idea what vortex methods are about. The complete derivations and convergence proofs are tedious and lengthy. For more information, the author recommends the book Cottet and Koumoutsakos [1]. In this text, we follow the method described by Gharakhani and Stock [2].

## 2 Description of the Method

In this section, the main idea of vortex methods is outlined. We begin with the vorticity transport equation, an equivalent formulation of the Navier-Stokes equations in terms of vorticity. This is advantageous because—unlike velocity—vorticity usually has only local support. Furthermore the pressure term disappears from the equations. While vortex methods for viscid flows do exist, we do not cover them here, as their treatment would exceed the scope of this document.

In the next step we introduce the particle approximation and describe how to obtain the velocity field from the vorticity. In order to treat boundary conditions, we introduce the Helmholtz decomposition and present a reformulation of the resulting Poisson problem in terms of a boundary integral equation. In order to solve this equation, we propose a simple boundary element discretisation.

## 2.1 Material Derivative

Before stating the vorticity transport equation, we briefly recall the meaning of the material derivative.

**Definition 1.** (Material Derivative) *The material derivative is defined as the time rate of change of some property* $(\bullet)$ *for a material element subjected to a velocity field* $\mathbf{u}$*. In a Lagrangian frame of reference* $(\boldsymbol{\chi}, t)$ *it corresponds to the partial derivative with respect to time:*

$$\frac{\mathrm{D}(\bullet)}{\mathrm{D}t} := \frac{\partial(\bullet)}{\partial t}.$$

*In an Eulerian frame of reference* $(\mathbf{x}, t)$ *we get the following, probably more well known expression:*

$$\frac{\mathrm{D}(\bullet)}{\mathrm{D}t} = \left(\frac{\partial(\bullet)}{\partial t}\right)_{\boldsymbol{\chi}=\text{const.}} = \frac{\partial(\bullet)}{\partial t} + (\mathbf{u} \cdot \nabla)(\bullet).$$

This can, for example, be used to abbreviate the momentum equation of the incompressible Navier-Stokes equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) \equiv \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \eta\Delta\mathbf{u}. \tag{1}$$

More importantly, it corresponds to the *ordinary time derivative* when tracing a fluid particle.

## 2.2 Problem Description

Before giving the vorticity transport equation (VTE), we introduce the notion of vorticity and vectorial circulation.

**Definition 2.** (Vorticity and Vectorial Circulation) *The vorticity* $\boldsymbol{\omega}$ *of a velocity field* $\mathbf{u}$ *is given by:*

$$\boldsymbol{\omega} := \nabla \times \mathbf{u}.$$

*The vectorial circulation* $\boldsymbol{\Gamma}$ *of some fluid volume* $V$ *is defined as:*

$$\boldsymbol{\Gamma} := \int_V \boldsymbol{\omega}\,\mathrm{d}x.$$

*Therefore, vorticity can be interpreted as vectorial circulation per unit volume.*

Note that vectorial circulation and 'ordinary' circulation are different concepts. The name is somewhat misleading, however we use it here as it is also used in the article by Gharakhani and Stock [2].

The VTE is a reformulation of the momentum equation (1) in terms of vorticity. After taking the curl of the momentum equation, it is obtained using several identities from vector calculus in combination with the continuity equation $\nabla \cdot \mathbf{u} = 0$:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla\mathbf{u} + \nu\Delta\boldsymbol{\omega}. \tag{2}$$

Here, we have set $\nu := \eta/\rho$. A complete derivation can, e. g., be found in the book of Schröder [9], pp. 196–207. Note that the pressure term has disappeared from this equation.

The first term on the right hand side is called *vortex stretching* and plays a key role in three-dimensional flows. It describes how vortices are stretched and deformed. In a two-dimensional flow, this term is identically zero. If we additionally neglect the viscous term, the right hand side is identically zero, meaning that no vorticity is created or destroyed inside the flow field. Two-dimensional, inviscid flows are therefore never turbulent. For this case detailed and rigorous convergence proofs for vortex methods exist [1].

The VTE in combination with the continuity equation forms a complete, equivalent formulation of the Navier-Stokes equations. As mentioned before, we do not treat viscosity in this text. In the following, we will apply a vortex method to an example problem.

**Definition 3.** (EXAMPLE PROBLEM) *Let $\Omega := \mathbb{R}^3 \setminus S_1(\mathbf{0})$ or a 'disc' as in the referenced article [2]. Then solve for $t \in [0;T]$:*

$$\begin{cases} \dfrac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0,\ \nabla \times \mathbf{u} = \boldsymbol{\omega} & \text{in } \Omega, \end{cases}$$

*subject to the no-through-flow boundary condition and a given speed $\mathbf{u}_\infty$ at infinity:*

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega, \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}_\infty && \text{for } \mathbf{x} \to \infty, \end{aligned}$$

*and initial conditions:*

$$\boldsymbol{\omega}(\cdot, 0) = \boldsymbol{\omega}_0 \qquad \text{in } \Omega,$$

*where $\mathbf{n}$ denotes the surface normal pointing into the fluid domain.*

### 2.3 Particle Approximation

The main idea behind vortex methods is to approximate the *continuous vorticity field* by a set of *discrete particles*. A particle $p$ is a Dirac Delta distribution centred at position $\mathbf{x}_p$ and carrying vectorial circulation $\boldsymbol{\Gamma}_p$, giving rise to the following approximation:

$$\boldsymbol{\omega}(\mathbf{x}, t) \approx \sum_{p=1}^{N_p} \boldsymbol{\Gamma}_p(t) \delta(\mathbf{x} - \mathbf{x}_p(t)), \tag{3}$$

where $N_p$ is the number of particles. We assume that the particles are fluid elements which move corresponding to the local velocity:

$$\frac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}_p(t), t), \qquad p = 1, \ldots, N_p. \tag{4}$$

As mentioned before, by definition, the material derivative corresponds to the ordinary time derivative when tracing fluid elements. Therefore, the VTE (2) directly translates into:

$$\frac{\mathrm{d}\boldsymbol{\Gamma}_p(t)}{\mathrm{d}t} = \boldsymbol{\Gamma}_p(t) \cdot \nabla \mathbf{u}(\mathbf{x}_p(t), t), \qquad p = 1, \ldots, N_p. \tag{5}$$

Note that the VTE has now become a system of ordinary differential equations. In principle, these can be solved using classical time-stepping techniques such as Runge-Kutta or multistep methods. However, in order to do so, we still need to specify how to determine $\mathbf{u}$ and $\nabla\mathbf{u}$ from the vorticity field. This will be described in the next section.

### 2.4 Computation of the Velocity Field

The velocity field is determined by the remaining two partial differential equations:

$$\nabla \cdot \mathbf{u} = 0, \;\; \nabla \times \mathbf{u} = \boldsymbol{\omega}.$$

These equations have exactly the same form as Maxwell's equations for a steady-state magnetic field. Consequently, most of the following results are adoptions of theorems from that field. As we only want to give an introduction to vortex methods, a discussion of potential theory and electrodynamics is beyond the scope of this text. Here, we only give the required core results. An introduction to the topic as well as proofs for the results presented here can be found in Griffiths' book on electrodynamics [4].

The first important result is the so-called Helmholtz decomposition.

**Theorem 1.** (Helmholtz Decomposition) *The velocity* $\mathbf{u}$ *can uniquely be decomposed into three components:*

$$\mathbf{u} = \mathbf{u}_{\boldsymbol{\omega}} + \mathbf{u}_{\varphi} + \mathbf{u}_{\infty}.$$

*In this decomposition* $\mathbf{u}_\infty$ *is a constant describing the velocity at infinity. For internal flows it is identically zero.* $\mathbf{u}_\varphi$ *is a field satisfying* $\nabla \times \mathbf{u}_\varphi = \mathbf{0}$*, i. e., a potential flow, which vanishes at infinity. It describes the influence of boundary conditions. The remaining part* $\mathbf{u}_{\boldsymbol{\omega}}$ *is the velocity which is induced by the vorticity field and would result in the absence of any boundaries. It satisfies* $\nabla \times \mathbf{u}_{\boldsymbol{\omega}} = \boldsymbol{\omega}$*.*

Because $\mathbf{u}_\infty$ is part of the problem description, only two of these components are unknown. $\mathbf{u}_{\boldsymbol{\omega}}$ can be computed with the help of the Biot-Savart law:

**Theorem 2.** (Biot-Savart Law) *The velocity component* $\mathbf{u}_{\boldsymbol{\omega}}$ *is given by:*

$$\mathbf{u}_{\boldsymbol{\omega}}(\mathbf{x},t) = -\frac{1}{4\pi}\int_\Omega \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} \times \boldsymbol{\omega}(\mathbf{y},t)\,\mathrm{d}\mathbf{y}.$$

By inserting the particle approximation (3) into the Biot-Savart law, we immediately obtain:

$$\begin{aligned}
\mathbf{u}_{\boldsymbol{\omega}}(\mathbf{x},t) &= -\frac{1}{4\pi}\int_\Omega \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} \times \boldsymbol{\omega}(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \\
&\approx -\frac{1}{4\pi}\int_\Omega \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} \times \left(\sum_{p=1}^{N_p} \boldsymbol{\Gamma}_p(t)\delta(\mathbf{y}-\mathbf{x}_p(t))\right)\mathrm{d}\mathbf{y} \\
&= -\frac{1}{4\pi}\sum_{p=1}^{N_p} \frac{\mathbf{x}-\mathbf{x}_p(t)}{\|\mathbf{x}-\mathbf{x}_p(t)\|^3} \times \boldsymbol{\Gamma}_p(t).
\end{aligned} \tag{6}$$

Likewise, for the vortex stretching, the spatial velocity gradient $\nabla\mathbf{u}$ can be obtained by applying the $\nabla$-operator to the above formula.

As we need to evaluate the velocity field at every particle location, direct evaluation of the sum at all locations would lead to a time complexity of $\mathcal{O}(N_p^2)$. For a long time this made the method prohibitively expensive for all but the smallest simulations. Due to the invention of the fast multipole method (FMM) in the late 1980s by Greengard and Rokhlin [3], an $\mathcal{O}(N_p)$ algorithm became available and interest in vortex methods grew again.

The remaining unknown component of the velocity field is the potential flow $\mathbf{u}_\varphi$, which needs to be determined such that it vanishes at infinity and that the no-through-flow condition is fulfilled on the boundaries:

$$\begin{aligned} & \mathbf{u}\cdot\mathbf{n} &&= 0 \\ \iff & (\mathbf{u}_\omega + \mathbf{u}_\varphi + \mathbf{u}_\infty)\cdot\mathbf{n} &&= 0 \\ \iff & \mathbf{u}_\varphi\cdot\mathbf{n} &&= -(\mathbf{u}_\omega + \mathbf{u}_\infty)\cdot\mathbf{n}. \end{aligned} \tag{7}$$

Because $\mathbf{u}_\varphi$ is a potential flow, it is the gradient of a scalar function, i. e., there is a scalar function $\varphi$ such that we have $\mathbf{u}_\varphi = \nabla\varphi$. By inserting this into the continuity equation we obtain Laplace's equation:

$$\nabla\cdot\mathbf{u}_\varphi = 0 \iff \Delta\varphi = 0. \tag{8}$$

In the next section we describe how this problem is solved.

## 2.5 Boundary Element Method

As described in the last section, retrieving the velocity field requires solving Laplace's equation. There are numerous ways to do this. Finite Element Methods, for example, are robust and very fast. However, as they do require a domain mesh, their use would render the whole method mesh based again. Boundary Element Methods (BEM) are a mesh-free alternative. There are a variety of different BEM formulations, for their derivation and implementation the reader is referred to the books by Liu [5] and Sauter and Schwab [8]. Here, we can only give a *short motivation* of the methods. In this text we use the formulation given in the article by Gharakhani and Stock [2].

We begin the motivation by restating the Biot-Savart law:

$$\mathbf{u}_\omega(\mathbf{x},t) = -\frac{1}{4\pi}\int_\Omega \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\times\boldsymbol{\omega}(\mathbf{y},t)\,\mathrm{d}\mathbf{y}.$$

Now assume that the vorticity vanishes everywhere within the domain and only exists on the boundaries themselves, i. e.,

$$\operatorname{supp}\boldsymbol{\omega}\subset\partial\Omega.$$

In this case, the flow field described by the Biot-Savart law would fulfil $\nabla\times\mathbf{u} = 0$ *within* the fluid domain $\Omega$ and would hence be a potential flow. The idea is then to place vorticity on the boundaries in such a way that the boundary conditions are fulfilled. In order to distinguish the vorticity on the boundaries from the vorticity inside the domain, we call the former $\boldsymbol{\gamma}$, which is commonly called the *vortex sheet strength*. The question that then arises is: 'How to compute the vortex sheet strength?'

Unfortunately the Biot-Savart law is only valid at positions $\mathbf{x}$ *inside* the domain. In order to obtain an expression for the boundaries one lets $\mathbf{x}$ approach the boundaries in a limiting process. This limiting process is complicated as the fraction in the integral becomes singular on the boundaries. For the details the reader is referred to the literature. Here we just give the result:

$$\mathbf{u}(\mathbf{x},t)\big|_{\mathbf{x}\in\partial\Omega} = \frac{\boldsymbol{\gamma}(\mathbf{x},t)\times\mathbf{n}}{2} - \frac{1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\times\boldsymbol{\gamma}(\mathbf{y},t)\,\mathrm{d}\mathbf{y}. \tag{9}$$

The no-through-flow condition restricts the normal component of the velocity. One can show that this corresponds to a condition for the tangential components of the vorticity while the normal component is identically zero. The equation that $\boldsymbol{\gamma}$ needs to fulfil then becomes:

$$-(\mathbf{u}_{\boldsymbol{\omega}}+\mathbf{u}_\infty)\cdot\mathbf{t}(\mathbf{x}) = \left(\frac{\boldsymbol{\gamma}(\mathbf{x},t)\times\mathbf{n}(\mathbf{x})}{2} - \frac{1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\times\boldsymbol{\gamma}(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\right)\cdot\mathbf{t}(\mathbf{x}) \qquad \forall\mathbf{x}\in\partial\Omega. \tag{10}$$

Here, $\mathbf{t}(\mathbf{x})$ denotes an arbitrary unit vector which is tangential to the surface at $\mathbf{x}$. This is a *boundary integral equation* for the unknown vortex sheet strength $\boldsymbol{\gamma}$. It is a Fredholm equation of the second kind. In the following we will omit the tangential vector and the time argument in order to ease notation.

By defining the operator $\mathcal{L}$:

$$(\mathcal{L}\boldsymbol{\gamma})(\mathbf{x}) := \frac{\boldsymbol{\gamma}(\mathbf{x})\times\mathbf{n}(\mathbf{x})}{2} - \frac{1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\times\boldsymbol{\gamma}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{11}$$

this can be rewritten as an operator equation:

$$\mathcal{L}\boldsymbol{\gamma} = -(\mathbf{u}_{\boldsymbol{\omega}}+\mathbf{u}_\infty). \tag{12}$$

Gharakhani and Stock discretise this problem with approximating the boundaries by triangular elements $S_i, i=1,\ldots,N_T$, which in this context are commonly referred to as 'panels' or *boundary elements*. The vortex sheet is approximated by placing vortex particles carrying vectorial circulation $\boldsymbol{\Gamma}_i$ on the centre of each panel. They then use a Petrov-Galerkin approach and test against the space of piecewise constants, i. e., they require that the following holds:

$$\left\langle \mathcal{L}\left(\sum_{i=1}^{N_T}\boldsymbol{\Gamma}_i\delta(\mathbf{x}-\mathbf{x}_i)\right),\chi_{S_i}\right\rangle = -\langle\mathbf{u}_{\boldsymbol{\omega}}+\mathbf{u}_\infty,\chi_{S_i}\rangle \qquad i=1,\ldots,N_T. \tag{13}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the $L_2(\partial\Omega)$-inner product, $\mathbf{x}_i$ the geometric centre of panel $S_i$ and $\chi_{S_i}$ the characteristic function of the corresponding panel:

$$\chi_{S_i}:\partial\Omega\to\{0,1\}, \qquad \mathbf{x}\mapsto\begin{cases}1 & \text{if } \mathbf{x}\in S_i,\\ 0 & \text{else.}\end{cases} \tag{14}$$

A simple calculation shows that equation (13) is equivalent to:

$$\begin{aligned}
&\frac{\boldsymbol{\Gamma}_i\times\mathbf{n}_i}{2} - \frac{1}{4\pi}\int_{S_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{N_T}\frac{\mathbf{x}_j-\mathbf{y}}{\|\mathbf{x}_j-\mathbf{y}\|^3}\times\boldsymbol{\Gamma}_j\right)\mathrm{d}\mathbf{y} = \mathbf{Q}_i \qquad i=1,\ldots,N_T,\\
&\mathbf{Q}_i := -\int_{S_i}(\mathbf{u}_{\boldsymbol{\omega}}+\mathbf{u}_\infty)\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{15}$$

Here, $\mathbf{n}_i$ is the unit normal vector at position $\mathbf{x}_i$ pointing into $\Omega$. Note that in this particular discretisation there are no singular integrals involved and that ordinary quadrature rules can be used to evaluate the integrals. For each panel $S_i$, this vector equation is multiplied with the two local tangent vectors of that panel and a *linear system of equations* for the tangential components of the particle strengths is obtained. This system is well-conditioned, but due to the sum over all particles in the integral it is *densely populated*.

To avoid the storage complexity of $\mathcal{O}(N_T^2)$, the system is typically solved using iterative solvers such as GMRES. Also note that the sum under the integral has the same form as in equation (6). We can thus make use of the FMM to evaluate that sum on all panels in $\mathcal{O}(N_T)$ time, leading to an efficient solver for the unknown vortex sheet strength.

While in practice this discretisation seems to work well, unfortunately, neither the article by Gharakani and Stock [2], nor their referenced article [6] mention any rigorous convergence analysis.

### 2.6 Putting Everything Together

Now that we have described the necessary tools, we can finally describe the algorithm.

1. Create an initial particle field that approximates $\boldsymbol{\omega}_0$, e. g., by placing uniformly spaced particles into the support of $\boldsymbol{\omega}$ and by setting their vectorial circulation to the local value of $\boldsymbol{\omega}_0$.

2. Create an triangulation $\mathcal{S} = \{S_i\}$ of the boundaries and place *immovable* vortex particles on the triangles' centres.

3. Use a standard time-stepping technique, e. g., a Runge-Kutta method or a multistep method, for advancing the ODEs (4) and (5) in time. In order to evaluate the velocities and their gradients in each (sub-)step, do the following:
   a) Compute $\mathbf{u}_\omega$ at each particle location as well as at the quadrature points on the boundaries with the help of the Biot-Savart law (6).
   b) Compute $\nabla\mathbf{u}_\omega$ at each particle location, again by using the Biot-Savart law.
   c) Compute the unknown vortex sheet strength on the boundaries by solving equation (15).
   d) Now pretend the particles on the boundaries are ordinary particles. Use the Biot-Savart law (6) in order to compute $\mathbf{u}_\varphi$ and $\nabla\mathbf{u}_\varphi$ at each particle inside the domain.

4. Remove any particles that might have escaped the fluid domain. This should only rarely happen if the surface triangulation is sufficiently refined.

5. Repeat steps 3 and 4 until the requested termination time is reached.

## 3 Outlook

In the previous section we described a very basic vortex method for incompressible, inviscid flow problems. In this section we first want to highlight some of the beneficial properties of the method:

- Only a mesh of the boundaries is needed.

- The method can be implemented in such a way that the overall circulation is conserved.

- Due to the mesh-less nature of the scheme, there is nothing like a CFL-condition.
- When using the Fast Multipole Method to evaluate the velocities, the overall time and space complexities are $\mathcal{O}(N_p + N_{\text{iter}} N_T)$. Here $N_{\text{iter}}$ denotes the number of iterations for the linear solver. As the system is well-conditioned, it can be assumed to be a small constant. This means that the method offers optimal complexities.

Gharakhani and Stock also implemented an extension to include viscous effects in their calculations. As the support of the vorticity typically grows during the course of the simulation, they continuously insert new particles into those areas where there are not enough particles to describe the features of the vorticity field.

In practice one often uses so-called 'mollified' particles, where one replaces the Dirac delta distributions with smooth approximations thereof, e. g., by the density function of the normal distribution. The particles are then often called *vortex blobs*. As can be shown, this removes the singularity from the Biot-Savart law. The width of the bell curve (the standard deviation of the normal distribution) then controls up to which scale the velocity field is resolved, giving the simulation a LES-like character.

Vortex methods continue to be an active area of research, the current trend is to move the FMM computations on GPUs [7].

# References


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