Modeling Unsteady Flow in Turbomachinery Using a Harmonic Balance Technique

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30.04.2014

Master seminar CES WS 2013/2014
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1 Introduction

Computational Fluid Dynamics has become one of the most important tools to design new turbomachinery and there are numerous methods available depending on the specific requests.

Since direct numerical simulations (DNS) and large eddy simulations (LES) are still way to expensive regarding computational costs, unsteady Reynolds-Averaged Navier-Stokes (URANS) techniques have been the most efficient ones to meet the industrial needs of today.

But as large scales of turbomachinery flows could be considered periodic in time, conventional unsteady techniques (see chapter 3.1) do not deliver enough computational efficiency simulating these flows. This is due to the fact, that these common techniques do not account for periodicity, which leads to a major loss in computational time, as a transient regime must be by-passed before the flow is periodic.

Taking this into account, new techniques, based on the assumption of time-periodic flows, have been developed recently.

One of these new approaches is the Harmonic Balance method which is presented in this term paper.

2 Governing Equations

2.1 Navier-Stokes-Fourier equations

The viscous compressible three-dimensional system of Navier-Stokes-Fourier equations can be written in vector notation

$$\frac{\partial Q}{\partial t} + \nabla[F - F_v] = S \qquad (1)$$

with the vector of conservation variables

$$Q = [\rho \quad \rho u \quad \rho v \quad \rho w \quad \rho E]^T,$$

the non-viscous and the viscous flux vectors, respectively

$$F = \begin{bmatrix} \rho C \\ \rho u C + n_x p \\ \rho v C + n_y p \\ \rho w C + n_z p \\ \rho E C + C p \end{bmatrix}, \qquad F_v = \begin{bmatrix} 0 \\ n_x \tau_{xx} + n_y \tau_{xy} + n_z \tau_{xz} \\ n_x \tau_{yx} + n_y \tau_{yy} + n_z \tau_{yz} \\ n_x \tau_{zx} + n_y \tau_{zy} + n_z \tau_{zz} \\ n_x \theta_x + n_y \theta_y + n_z \theta_z \end{bmatrix}$$

and the source term vector

$$S = [0 \quad \rho f_x \quad \rho f_y \quad \rho f_z \quad \rho f \cdot c + \dot{q}]^T.$$

The system of equations consists of the conservation of mass, momentum in x, y, z direction and energy, where ρ denotes the density, $c = [u \quad v \quad w]^T$ the velocity in x, y and z direction, respectively, E the specific total energy, p the pressure, $n = [n_x \quad n_y \quad n_z]^T$ the normal vector and $f = [f_x \quad f_y \quad f_z]^T$ external mass forces.

The deviatoric stress tensor τ reads as

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}.$$

F contains the contravariant velocity

$$C = c \cdot n = n_x u + n_y v + n_z w$$

and F_v the abbreviations

$$\theta_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + \lambda \frac{\partial T}{\partial x},$$

$$\theta_y = u\tau_{yx} + v\tau_{yy} + w\tau_{yz} + \lambda \frac{\partial T}{\partial y},$$

$$\theta_z = u\tau_{zx} + v\tau_{zy} + w\tau_{zz} + \lambda \frac{\partial T}{\partial z},$$

where $\lambda \nabla T$ is the heat conduction given by Fourier's law.

2.2 Closures for the Navier-Stokes-Fourier equations

The Navier-Stokes-Fourier equations represent five coupled partial differential equations for the conservation variables shown in Q with the unknown quantities ρ, c, e, p, T. Therefore two more equations must be given to model the thermodynamic dependencies between the conservation variables.

In an ideal gas, the pressure p can be expressed as function of temperature T and density ρ (ideal gas law/thermal equation of state)

$$p = \rho R T. \tag{2}$$

With the specific internal energy

$$e = c_v T, \tag{3}$$

and the definition of the ideal gas constant R and the adiabatic index κ, respectively,

$$R = c_p - c_v, \tag{4}$$

$$\kappa = \frac{c_p}{c_v}, \tag{5}$$

eqn. (2) can be rearranged as

$$p = \rho(\kappa - 1)e. \tag{6}$$

Using eqn. (6) and the definition of the specific total energy

$$E = e + \frac{u^2 + v^2 + w^2}{2}, \tag{7}$$

the specific total energy can be rewritten as a function of the pressure, density and velocity $c = [u \quad v \quad w]^T$

$$E = \frac{p}{(\kappa - 1)\rho} + \frac{u^2 + v^2 + w^2}{2}. \tag{8}$$

In addition to the thermodynamic closure, some material specific dependencies must be provided as to close the stresses as well as the heat flux.

The deviatoric stress tensor reads with Stoke's law as

$$\tau = 2\vartheta \left[\frac{1}{2}(\nabla c) + \frac{1}{2}(\nabla c)^T\right] + \mu[(\nabla \cdot c)\delta] \tag{9}$$

where ϑ denotes the kinematic viscosity, μ the dynamic viscosity and δ the Kronecker delta. The stress deviator has zero trace, is symmetric and therefore its components can be written as

$$\tau_{xx} = 2\mu \left[\frac{\partial u}{\partial x} - \frac{1}{3}(\nabla \cdot c)\right],$$

$$\tau_{yy} = 2\mu \left[\frac{\partial v}{\partial y} - \frac{1}{3}(\nabla \cdot c)\right],$$

$$\tau_{zz} = 2\mu \left[\frac{\partial w}{\partial z} - \frac{1}{3}(\nabla \cdot c)\right],$$

$$\tau_{xy} = \mu \left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right],$$

$$\tau_{xz} = \mu \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right],$$

$$\tau_{yz} = \mu \left[\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right].$$

The viscosity coefficient can be determined using, for example, the Sutherland model

$$\mu = \mu_{ref} \frac{T_{ref} + C_S}{T + C_S} \left[\frac{T}{T_{ref}}\right]^{\frac{3}{2}} \tag{10}$$

with the Sutherland constant C_S, the viscosity of reference μ_{ref} and the temperature of reference T_{ref}, respectively.

The material's conductivity λ from Fourier's law shows a similar dependency regarding the temperature and can therefore be modeled using the viscosity μ and Prandtl number Pr

$$\lambda = c_p \frac{\mu}{Pr}. \tag{11}$$

2.3 Euler equations

The Euler equations are a special case of inviscid flow. The equations represent conservation of mass (continuity), momentum and energy, corresponding to the Navier–Stokes-Fourier equations with vanishing viscosity and no heat conduction terms.

In this context, they are not explicitly written down as they can be derived in a straight forward way from the set of Navier-Stokes equations (1). Nevertheless, the example given in chapter 3.2 is shown using a set of two-dimensional Euler equations.

The Euler equations can be closed using the same equations derived in chapter 2.2.

2.4 Reynolds-Averaged Navier-Stokes equations (RANS)

In most cases, the regarded flows are mathematically difficult to model (e.g. highly turbulent, multiphase flow and/or combustion). As in these flows the Navier-Stokes equations with industrially relevant Reynolds numbers cannot be calculated numerically at a reasonable computational cost, the so called Reynolds-averaged Navier–Stokes equations (RANS) are introduced.

In this approach time-averaged equations of conservation (mass, momentum and energy) are developed in order to give approximate time-averaged solutions to the Navier–Stokes equations.

The RANS equations are derived using the Reynolds decomposition, whereby an instantaneous quantity is decomposed into its time-averaged $(\bar{\phi})$ and fluctuating $(\tilde{\phi})$ quantities. This decomposition writes as

$$\phi = \bar{\phi} + \phi' \tag{12}$$

with

$$\bar{\phi} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \phi \, dt$$

and the following governing rules

$$\overline{\phi'} = 0, \qquad \bar{\bar{\phi}} = \bar{\phi}, \qquad \overline{\phi_1 + \phi_2} = \overline{\phi_1} + \overline{\phi_2}, \qquad \overline{\phi_1 \phi_2} = \overline{\phi_1}\,\overline{\phi_2}. \tag{13}$$

Inserting ansatz (12) into the equations of conservation yields a large amount of unknown correlations between the density and other fluctuating quantities (Note: this applies only for the compressible case, in the incompressible case, they do not appear). To avoid these correlations, density weighted time averaging (Favre averaging) is often used additionally.

Favre averaging writes

$$\phi = \tilde{\phi} + \phi'' \tag{14}$$

with

$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}}, \tag{15}$$

where the overbars (e.g. $\overline{\rho\phi}$) denote averages using the Reynolds decomposition.

With eqn. (12) and eqn. (14) the equations of conservation for mass, momentum and energy hold

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial\bar{\rho}\tilde{c}_i}{\partial x_i} = 0, \tag{16}$$

$$\frac{\partial\bar{\rho}\tilde{c}_i}{\partial t} + \frac{\partial\bar{\rho}\tilde{c}_i\tilde{c}_j}{\partial x_j} + \frac{\partial\bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\bar{\tau}_{ij} + \bar{\rho}\widetilde{c_i''c_j''}\right) = 0, \tag{17}$$

$$\begin{aligned}&\frac{\partial\bar{\rho}\tilde{E}}{\partial t} + \frac{\partial\bar{\rho}\tilde{c}_i\tilde{E}}{\partial x_i} + \frac{\partial\tilde{c}_i\bar{p}}{\partial x_i}\\ &+ \frac{\partial}{\partial x_i}\left(-\tilde{c}_j\bar{\tau}_{ij} + \bar{\rho}\tilde{c}_j\widetilde{c_i''c_j''} - \overline{c_j''\tau_{ij}} + \overline{\rho c_j''\frac{1}{2}c_i''c_i''} + \bar{\rho}\widetilde{c_i''h''} - \lambda\bar{T}\right) = 0\end{aligned} \tag{18}$$

with

$$\tilde{E} = \tilde{e} + \frac{\tilde{c}_i\tilde{c}_i}{2} + k$$

and the turbulent energy

$$k = \frac{\widetilde{c_i''c_i''}}{2}.$$

The viscous stress tensor can be extended with the Reynolds stress tensor

$$\bar{\tau}_{t,ij} = -\bar{\rho}\widetilde{c_i''c_j''} \tag{19}$$

and analogously, the diffusive heat flow $-\lambda\nabla T$ with the turbulent heat flux

$$\dot{q}_{t,i} = -\bar{\rho}\widetilde{c_i''h''}. \tag{20}$$

The set of equations derived above (16) - (18) is usually called Reynolds-averaged Navier-Stokes equations (RANS) [1]. Nevertheless, there is also literature [2], where equations both Reynolds- and Favre-averaged are referred to as Favre-averaged Navier-Stokes (FANS). However, in this very scope, RANS defines the Reynolds- as well as Favre-averaged equations of conservation.

The closures given in chapter 2.2 are still valid except for the stress tensor, the heat flow and the newly added turbulent energy, which must now be closed using turbulence models (e.g. Prandtl's mixing-length concept, eddy viscosity, sub-grid scale eddy viscosity, $k-\varepsilon$ −model, $k-\omega$ −model).

3 State Of The Art

The governing methods for aerodynamic analysis of unsteady flows in turbomachinery have been the non-linear time-domain analysis and the time-linearized frequency-domain analysis.

3.1 Non-linear time-domain analysis

As this method is the classic way of computational fluid dynamics, there is just a brief introduction given. Nevertheless, more information regarding the use of the non-linear time-domain analyses can be found in, for example, references [3] - [5].

In this approach, the first step is to discretize the fluid equations (see chapters 2.1 and 2.2) on a spatial grid. This can be achieved using finite differences (FDM), finite volumes (FVM) or finite elements (FEM). The computed solution is then marched with defined time steps from one time level to the next using common computational fluid dynamic schemes. There are various schemes applicable depending on the chosen fluid equations and the discretization method. They are usually classified into central (e.g. Lax-Wendroff method, Runge-Kutta method) or upwind (e.g. Godunovs scheme) schemes and implicit or explicit schemes. The resulting large and sparse matrices are solved by stationary iterative solvers (e.g. Jacobi method, Gauss-Seidel method, Successive over-relaxation method) or Krylov subspace methods (e.g. conjugate gradient method, MINRES, GMRES).

The time-domain method is rather straightforward to implement and to work with, respectively and is also capable of modeling both linear and nonlinear unsteady flow. Unfortunately, due to the required stability of the code, the achievable step size, especially in explicit schemes, is pretty small and therefore leads to large computational costs and small computational speed.

3.2 Time-linearized frequency-domain analysis

To avoid these limited step size in order to increase the computational speed, the following time-linearized method is presented. The basic idea behind this method is to transfer the time-dependent equations into time-independent equations resulting in time-linearized equations very easy to solve.

As this approach is not as intuitive as the previous one, this method is shown on the example of the 2-dimensional Euler equations.

The unsteady and non-linear Euler equations derived from the Navier-Stokes equations in eqn. (1) read in two dimensions and vector notation

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \qquad (21)$$

with

$$U = \left[\rho \quad \rho u \quad \rho v \quad \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2)\right]^T,$$

$$F = \left[\rho u \quad \rho u^2 + p \quad \rho uv \quad \frac{\gamma}{\gamma - 1} pu + \frac{1}{2}\rho u(u^2 + v^2)\right]^T,$$

$$G = \left[\rho v \quad \rho uv \quad \rho v^2 + p \quad \frac{\gamma}{\gamma - 1} pv + \frac{1}{2}\rho v(u^2 + v^2)\right]^T.$$

Assuming the amplitude of the unsteadiness in the fluid is quite small, the equations can be decomposed into a sum of a steady flow component (time-independent) and an unsteady flow component (time-dependent)

$$\rho(x, y, t) = \bar{\rho}(x, y) + \tilde{\rho}(x, y, t), \qquad (22)$$

$$u(x, y, t) = \bar{u}(x, y) + \tilde{u}(x, y, t), \qquad (23)$$

$$v(x, y, t) = \bar{v}(x, y) + \tilde{v}(x, y, t), \qquad (24)$$

$$p(x, y, t) = \bar{p}(x, y) + \tilde{p}(x, y, t). \qquad (25)$$

Note that these assumption is only valid for flows where the disturbance is less than 10% of the steady mean flow [6]. Inserting eqns. (22) - (25) in eqn. (21), neglecting terms of second and higher order and collecting by zeroth- and first-order leads on the one hand to the steady state zeroth-order equations

$$\frac{\partial \bar{F}}{\partial x} + \frac{\partial \bar{G}}{\partial y} = 0, \qquad (26)$$

where

$$\bar{F} = \left[\bar{\rho}\bar{u} \quad \bar{\rho}\bar{u}^2 + \bar{p} \quad \bar{\rho}\bar{u}\bar{v} \quad \frac{\gamma}{\gamma - 1}\bar{p}\bar{u} + \frac{1}{2}\bar{\rho}\bar{u}(\bar{u}^2 + \bar{v}^2)\right]^T,$$

$$\bar{G} = \left[\bar{\rho}\bar{v} \quad \bar{\rho}\bar{u}\bar{v} \quad \bar{\rho}\bar{v}^2 + \bar{p} \quad \frac{\gamma}{\gamma - 1}\bar{p}\bar{v} + \frac{1}{2}\bar{\rho}\bar{v}(\bar{u}^2 + \bar{v}^2)\right]^T,$$

which are the common Euler equations but with steady flow components only and therefore with no time-derivatives. On the other hand this leads to the unsteady first order equations

$$\frac{\partial(B_1\tilde{U})}{\partial t} + \frac{\partial(B_2\tilde{U})}{\partial x} + \frac{\partial(B_3\tilde{U})}{\partial y} = 0, \qquad (27)$$

where

$$\tilde{U} = [\tilde{\rho} \quad \tilde{u} \quad \tilde{v} \quad \tilde{p}]^T,$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{u} & \bar{\rho} & 0 & 0 \\ \bar{v} & 0 & \bar{\rho} & 0 \\ \frac{1}{2}(\bar{u}^2 + \bar{v}^2) & \bar{\rho}\bar{u} & \bar{\rho}\bar{v} & \frac{1}{\gamma - 1} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ \bar{u}^2 & 2\bar{\rho}\bar{u} & 0 & 1 \\ \bar{u}\bar{v} & \bar{\rho}\bar{v} & \bar{\rho}\bar{u} & 0 \\ \frac{1}{2}\bar{u}(\bar{u}^2 + \bar{v}^2) & \frac{\gamma}{\gamma - 1}\bar{p} + \frac{3}{2}\bar{\rho}\bar{u}^2 + \frac{1}{2}\bar{\rho}\bar{v}^2 & \bar{\rho}\bar{u}\bar{v} & \frac{\gamma\bar{u}}{\gamma - 1} \end{bmatrix},$$

$$B_3 = \begin{bmatrix} \bar{v} & 0 & \bar{\rho} & 0 \\ \bar{u}\bar{v} & \bar{\rho}\bar{v} & \bar{\rho}\bar{u} & 0 \\ \bar{v}^2 & 0 & 2\bar{\rho}\bar{v} & 1 \\ \frac{1}{2}\bar{u}(\bar{u}^2 + \bar{v}^2) & \bar{\rho}\bar{u}\bar{v} & \frac{\gamma}{\gamma - 1}\bar{p} + \frac{1}{2}\bar{\rho}\bar{u}^2 + \frac{3}{2}\bar{\rho}\bar{v}^2 & \frac{\gamma\bar{v}}{\gamma - 1} \end{bmatrix}.$$

The mean flow equations (26) are non-linear in the steady state primitive variables and the solution is known from the steady state Euler equations, whereas the unsteady equations (27) are non-linear in the steady state variables but linear in the perturbation variables. Since the steady state primitive variables are known from eqn. (26), the unsteady equations are linear.

As many of the unsteady flow effects of interest are periodic in time, the perturbation variables are substituted with a Fourier-Series ansatz

$$\phi(x,y,t) \rightarrow \phi(x,y)e^{j\omega t} \qquad (28)$$

with $\omega = 2\pi/T$. Hence, the unsteady equations now read

$$j\omega B_1\widetilde{U} + \frac{\partial(B_2\widetilde{U})}{\partial x} + \frac{\partial(B_3\widetilde{U})}{\partial y} = 0 \qquad (29)$$

and are therefore converted from time-dependent into time-independent. This set of equations can now be evaluated for every temporal frequency of interest (e.g. low engine order excitation caused by non-uniformites in the flow around the annulus due to differences in nominal identical vanes or upstream vane excitation and its effect on high pressure turbine blading).

The resulting equations can now be discretisized and solved the same way as presented in the previous model and fortunately they are not limited by small time-steps any more, which leads to inexpensive computational costs and fast results, respectively. Unfortunately, in this approach one is not capable of modeling dynamic non-linearities and per computational session, there are the periodic effects of only one frequency available.

More information regarding this method and possible applications can be found in references [6]-[8].

4 Harmonic Balance Technique

The Harmonic Balance Technique has been mainly developed at Duke University [9] and Stanford University [10], [11] under the name Time Spectral Method.

This approach is based on the assumption, that the flow in turbomachinery is periodic and thus, the set of conservation variables can be represented by Fourier coefficients. With these Fourier expressions, the periodic unsteady RANS (URANS) computation is decomposed in several coupled steady flow computations. Unfortunately, the method in its "base" form (Monofrequential Harmonic Balance Technique) has one major disadvantage; only one frequency and its harmonics can be considered.

It is well known, that dominant frequencies seen by a blade row are created by the passing of the neighboring rows. In a single stage turbine setup (rotor row following a stator row), the stator row sees the blade passing frequency and its harmonics of the rotor row whereas the rotor row resolves the vane passing frequency and its harmonics of the stator row. In this case, each row resolves only one frequency and its harmonics.

In the way more complex multistage setup, the row of interest perceives the passing frequencies of all the neighboring rows and as the number of blades of the neighbors usually differ, the passing frequencies are different, too. In this situation there is not only one frequency but several of them to solve and the Monofrequential Harmonic Balance Technique does not deliver the desired results.

Taking this into account, Ekici and Hall [12] presented the Multifrequential Harmonic Balance Technique.

4.1 Monofrequential Harmonic Balance Technique

The Navier-Stokes-Fourier equations (1) in integral form are given by

$$\frac{\partial}{\partial t}\int_{\Omega} Q dV + \int_{\partial\Omega} [F - F_V] dA - \int_{\Omega} S\, dV = 0, \qquad (30)$$

which can be rewritten in the semi-discrete form

$$V\frac{\partial Q}{\partial t} + R(Q) = 0, \qquad (31)$$

where V denotes the control volume and $R(Q)$ denotes the residual vector resulting from spatial only discretization of the convective F and viscous F_V fluxes, respectively, and the source term S.

The first step in the Harmonic Balance Technique is the decomposition of the conservation variables into the discrete Fourier series with the angular frequency ω

$$\omega = \frac{2\pi}{T}.$$

The discrete Fourier series of a function f at the order of N reads

$$f_N(t) = \sum_{k=-N}^{N} \hat{f}_k exp\left(2i\pi n \frac{t}{T}\right), \tag{32}$$

where

$$\hat{f}_k = \frac{1}{2N+1} \sum_{n=0}^{2N} f_n exp\left(-2i\pi \frac{nk}{2N+1}\right), \qquad f_n = f\left(\frac{nT}{2N+1}\right).$$

Hence, the conservative variables can be written with the Fourier coefficients $\hat{Q}_k$ and $\hat{R}_k$

$$Q \to \sum_{k=-N}^{N} \hat{Q}_k exp(i\omega kt), \tag{33}$$

$$R(Q) \to \sum_{k=-N}^{N} \hat{R}_k exp(i\omega kt), \tag{34}$$

where N denotes the number of harmonics and k the mode number.

Inserting (33) and (34) into (31) yields

$$\sum_{k=-N}^{N} \left(i\omega kV \hat{Q}_{eq,k} + \hat{R}_{eq,k}\right) exp(i\omega kt) = 0, \tag{35}$$

$$0 \leq eq \leq number\ of\ equations,$$

where eq denotes the equation (mass, momentum, energy and e.g. turbulence transport equations).

The starting point of the resolution is chosen using the orthogonality of the complex function. That is, eqn. (35) is only valid, if

$$i\omega kV \hat{Q}_{eq,k} + \hat{R}_{eq,k} = 0 \tag{36}$$

holds.

With the conservative variables

$$\rho = \sum_{k=-N}^{N} \mathrm{P}_k exp(i\omega kt),$$

$$\rho u = \sum_{k=-N}^{N} U_k exp(i\omega kt),$$

$$\rho v = \sum_{k=-N}^{N} V_k exp(i\omega kt),$$

$$\rho w = \sum_{k=-N}^{N} W_k exp(i\omega kt),$$

$$\rho E = \sum_{k=-N}^{N} E_k exp(i\omega kt)$$

eqn. (36) can be rewritten as

$$\hat{R} + Vi\omega K_{mono}\hat{Q} = 0 \qquad (37)$$

with

$$\hat{R} = [R_{1,-N} \quad \cdots \quad R_{1,N} \quad R_{2,-N} \quad \cdots \quad R_{2,N} \quad R_{3,-N} \quad \cdots \quad R_{3,N} \\ R_{4,-N} \quad \cdots \quad R_{4,N} \quad R_{5,-N} \quad \cdots \quad R_{5,N}]^T,$$

$$K_{mono} = diag(-N \quad \cdots \quad N \quad -N \quad \cdots \quad N \quad -N \quad \cdots \quad N \quad -N \quad \cdots \quad N \quad -N \quad \cdots \quad N),$$

$$\hat{Q} = [\mathrm{P}_{-N} \quad \cdots \quad \mathrm{P}_N \quad U_{-N} \quad \cdots \quad U_N \quad V_{-N} \quad \cdots \quad V_N \quad W_{-N} \quad \cdots \quad W_N \quad E_{-N} \quad \cdots \quad E_N]^T.$$

In a further step the Fourier transformation is introduced. To start with, the discrete Fourier transform (DFT) operator/matrix is defined as

$$B^* = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_{2N+1}^{1} & \omega_{2N+1}^{2} & \cdots & \omega_{2N+1}^{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_{2N+1}^{(2N)} & \omega_{2N+1}^{2(2N)} & \cdots & \omega_{2N+1}^{2N(2N)} \end{bmatrix}, \qquad (38)$$

where

$$\omega_{2N+1}^{k} = \omega_{2N+1}^{k+2N+1} = exp\left(-2i\pi \frac{k}{2N+1}\right).$$

The matrix B^* is easy invertible and its inverse writes

$$(B^*)^{-1} = \frac{1}{2N+1} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_{2N+1}^{-1} & \omega_{2N+1}^{-2} & \cdots & \omega_{2N+1}^{-2N} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_{2N+1}^{-(2N)} & \omega_{2N+1}^{-2(2N)} & \cdots & \omega_{2N+1}^{-2N(2N)} \end{bmatrix}. \qquad (39)$$

As the aim is to obtain the conservative variables at the 2N+1 instants equally spaced over one period T the conservative fields hold at the discrete points

$$\tilde{Q} = \begin{cases} \mathrm{P}^{\mathrm{i}} & if & 0 & \leqq i \leqq & (2N+1)-1 \\ U^{i-(2N+1)} & if & (2N+1) & \leqq i \leqq & 2(2N+1)-1 \\ V^{i-2(2N+1)} & if & 2(2N+1) & \leqq i \leqq & 3(2N+1)-1, \\ W^{i-3(2N+1)} & if & 3(2N+1) & \leqq i \leqq & 4(2N+1)-1 \\ E^{i-4(2N+1)} & if & 4(2N+1) & \leqq i \leqq & 5(2N+1)-1 \end{cases}$$

$$\tilde{R} = \begin{cases} R^{i} & if & 0 & \leqq i \leqq & (2N+1)-1 \\ R^{i-(2N+1)} & if & (2N+1) & \leqq i \leqq & 2(2N+1)-1 \\ R^{i-2(2N+1)} & if & 2(2N+1) & \leqq i \leqq & 3(2N+1)-1, \\ R^{i-3(2N+1)} & if & 3(2N+1) & \leqq i \leqq & 4(2N+1)-1 \\ R^{i-4(2N+1)} & if & 4(2N+1) & \leqq i \leqq & 5(2N+1)-1 \end{cases}$$

$$B = diag(B^* \quad B^* \quad B^* \quad B^* \quad B^*)$$

and $\hat{Q}$ and $\hat{R}$, respectively, can now be (new) defined as

$$\hat{Q} = \frac{1}{2N+1} B\tilde{Q},$$

$$\hat{R} = \frac{1}{2N+1} B\tilde{R}.$$

Using the previous definitions eqn. (37) can either be written

$$\hat{R} + Vi\omega K_{mono}\hat{Q} = 0 \tag{40}$$

with

$$\begin{aligned} \hat{R} = [&R_{1,0} \ \cdots \ R_{1,N} \ R_{1,-N} \ \cdots \ R_{1,-1} \ R_{2,0} \ \cdots \ R_{2,N} \ R_{2,-N} \ \cdots \ R_{2,-1} \\ &R_{3,0} \ \cdots \ R_{3,N} \ R_{3,-N} \ \cdots \ R_{3,-1} \ R_{4,0} \ \cdots \ R_{4,N} \ R_{4,-N} \ \cdots \ R_{4,-1} \\ &R_{5,0} \ \cdots \ R_{5,N} \ R_{5,-N} \ \cdots \ R_{5,-1}]^T, \end{aligned}$$

$$K = diag(\mathrm{N}^* \quad \mathrm{N}^* \quad \mathrm{N}^* \quad \mathrm{N}^* \quad \mathrm{N}^*), \qquad N^* = diag(0 \quad \cdots \quad N \quad -N \quad \cdots \quad -1),$$

$$\begin{aligned} \hat{Q} = [&\mathrm{P}_0 \ \cdots \ \mathrm{P}_N \ \mathrm{P}_{-N} \ \cdots \ \mathrm{P}_{-1} \ U_0 \ \cdots \ U_N \ U_{-N} \ \cdots \ U_{-1} \\ &V_0 \ \cdots \ V_N \ V_{-N} \ \cdots \ V_{-1} \ W_0 \ \cdots \ W_N \ W_{-N} \ \cdots \ W_{-1} \\ &E_0 \ \cdots \ E_N \ E_{-N} \ \cdots \ E_{-1}]^T \end{aligned}$$

or

$$\frac{1}{2N+1} B\tilde{R} + \frac{1}{2N+1} Vi\omega K_{mono} B\tilde{Q} = 0. \tag{41}$$

The change in the definitions of the matrices used in (40) compared to (37) is due to the definition of the discrete Fourier transform operator (38), that is, $\hat{R}$, K and $\hat{Q}$ are "resorted" to create an (easy) invertible DFT matrix. As B^* is invertible, B is obviously invertible, too

$$B^{-1} = diag((B^{*})^{-1} \quad (B^{*})^{-1} \quad (B^{*})^{-1} \quad (B^{*})^{-1} \quad (B^{*})^{-1}). \tag{42}$$

Eventually, the components of the Fourier transform can be transferred back into the time domain, which yields the equation

$$R(Q)^{*} + i\omega B^{-1} K_{mono} B Q^{*} = 0 \tag{43}$$

with

$$R(Q)^{*} = (2N+1) B^{-1} \hat{R},$$

$$Q^{*} = (2N+1) B^{-1} \hat{Q}.$$

4.2 Multifrequential Harmonic Balance Technique

In the most relevant turbomachinery practices, more than one frequency and its harmonics appear. Furthermore, in some cases the instants are not uniformly distributed over one period, requiring a different approach than the Monofrequential Harmonic Balance Technique. Ekici and Hall [12] have proposed a method solving the Navier-Stokes-Fourier equation with 3N+1 instants leading to a non-square discrete Fourier transformation (DFT) operator and thus, a pseudoinverse for this DFT operator must be found. Rauschenberger [13] has presented a way more convenient approach circumventing this problem and making his procedure similar to the Monofrequential Harmonic Balance Technique.

Recapitulating the previous chapter one begins with the Navier-Stokes-Fourier equation in the semi discrete form (eqn. (31))

$$V \frac{\partial Q}{\partial t} + R(Q) = 0,$$

which transferred to the frequency domain holds

$$\sum_{k=-N}^{N} \left(i\omega_k V \hat{Q}_{eq,k} + \hat{R}_{eq,k} \right) exp(i\omega_k t) = 0 \tag{44}$$

with $\omega_k = 2\pi f_k$.

Due to the fact that several frequencies appear being not multiples of each other (e.g. two or more base frequencies and their respective harmonics), eqn. (36) does not hold for the multifrequential ansatz. To ensure that the contribution of each ω_k is zero, that is, the following equation holds

$$i\omega_k V \hat{Q}_{eq,k} + \hat{R}_{eq,k} = 0, \tag{45}$$

the complex exponential form $exp(i\omega_k t_n)$ must form a basis. This can be achieved by choosing the instants in such a way that

$$A^* = \begin{pmatrix} exp(i\omega_{-N}t_0) & \cdots & exp(i\omega_0 t_0) & \cdots & exp(i\omega_N t_0) \\ \vdots & & \vdots & & \vdots \\ exp(i\omega_{-N}t_k) & \cdots & exp(i\omega_0 t_k) & \cdots & exp(i\omega_N t_k) \\ \vdots & & \vdots & & \vdots \\ exp(i\omega_{-N}t_N) & \cdots & exp(i\omega_0 t_N) & \cdots & exp(i\omega_N t_N) \end{pmatrix} \tag{46}$$

is invertible. The Matrix A^* of the size $n \times n$ is invertible if the following (equivalent) statements are satisfied

$$rank(A^*) = n,$$

$$det(A^*) \neq 0,$$

$$each\ eigenvalue \neq 0.$$

[13] gives a detailed description regarding the different approaches finding the instants that satisfy eqn. (45).

The conservation variables in the multifrequential ansatz can be written

$$\rho = \sum_{k=-N}^{N} \mathrm{P}_k exp(i\omega_k t),$$

$$\rho u = \sum_{k=-N}^{N} U_k exp(i\omega_k t),$$

$$\rho v = \sum_{k=-N}^{N} V_k exp(i\omega_k t),$$

$$\rho w = \sum_{k=-N}^{N} W_k exp(i\omega_k t),$$

$$\rho E = \sum_{k=-N}^{N} E_k exp(i\omega_k t),$$

which yields the Navier-Stokes equation

$$\hat{R} + ViK_{multi}\hat{Q} = 0 \tag{47}$$

with

$$\hat{R} = [R_{1,-N} \quad \cdots \quad R_{1,N} \quad R_{2,-N} \quad \cdots \quad R_{2,N} \quad R_{3,-N} \quad \cdots \quad R_{3,N} \\ R_{4,-N} \quad \cdots \quad R_{4,N} \quad R_{5,-N} \quad \cdots \quad R_{5,N}]^T,$$

$$K_{multi} = diag(K \quad K \quad K \quad K \quad K), \qquad K = diag(\omega_{-N} \quad \cdots \quad \omega_N),$$

$$\hat{Q} = [\mathrm{P}_{-N} \quad \cdots \quad \mathrm{P}_N \quad U_{-N} \quad \cdots \quad U_N \quad V_{-N} \quad \cdots \quad V_N \quad W_{-N} \quad \cdots \quad W_N \quad E_{-N} \quad \cdots \quad E_N]^T.$$

Transforming the conservative fields at the discrete points i into the frequency domain is accomplished by defining

$$\tilde{Q} = \begin{cases} \mathrm{P}^{\mathrm{i}} & if \quad 0 \leqq i \leqq (2N+1)-1 \\ U^{i-(2N+1)} & if \quad (2N+1) \leqq i \leqq 2(2N+1)-1 \\ V^{i-2(2N+1)} & if \quad 2(2N+1) \leqq i \leqq 3(2N+1)-1, \\ W^{i-3(2N+1)} & if \quad 3(2N+1) \leqq i \leqq 4(2N+1)-1 \\ E^{i-4(2N+1)} & if \quad 4(2N+1) \leqq i \leqq 5(2N+1)-1 \end{cases}$$

$$\tilde{R} = \begin{cases} R^{i} & if \quad 0 \leqq i \leqq (2N+1)-1 \\ R^{i-(2N+1)} & if \quad (2N+1) \leqq i \leqq 2(2N+1)-1 \\ R^{i-2(2N+1)} & if \quad 2(2N+1) \leqq i \leqq 3(2N+1)-1 \\ R^{i-3(2N+1)} & if \quad 3(2N+1) \leqq i \leqq 4(2N+1)-1 \\ R^{i-4(2N+1)} & if \quad 4(2N+1) \leqq i \leqq 5(2N+1)-1 \end{cases}$$

and

$$\hat{Q} = A^{-1}\tilde{Q},$$

$$\hat{R} = A^{-1}\tilde{R}$$

with

$$A^{-1} = diag((A^{*})^{-1} \quad (A^{*})^{-1} \quad (A^{*})^{-1} \quad (A^{*})^{-1} \quad (A^{*})^{-1}),$$

which results in the Navier Stokes equation in the frequency domain

$$A^{-1}\tilde{R} + \frac{1}{2N+1} ViA^{-1}\tilde{Q} = 0. \qquad (48)$$

Similarly to the monofrequential ansatz eqn. (48) can now be transformed back to the time domain using

$$Q^{*} = A\hat{Q},$$

$$R(Q)^{*} = A\hat{R},$$

which yields the final equation

$$R(Q)^{*} + ViAK_{multi}A^{-1}Q^{*} = 0. \qquad (49)$$

4.3 Numerical treatment

Monofrequential ansatz

In this context, two different approaches for the numerical treatment are briefly presented.

As Eqn. (43) is a steady equation (in the time domain), Hall et al. [9] propose to introduce a virtual time τ for numerical treatment in order to obtain a steady solution for each instant using a conventional CFD scheme resulting in the following equation

$$V\frac{\partial Q^*}{\partial \tau} + R(Q)^* + i\omega B^{-1}K_{mono}BQ^* = 0. \qquad (50)$$

The algorithm of McMullen et al. [14] makes use of the virtual time as well but in contrast to the previous approach, $R(Q)$ is calculated in the time domain and multiple systems

$$V\frac{\partial \hat{Q}_{eq,k}}{\partial \tau} + \hat{R}_{eq,k} + i\omega k V\hat{Q}_{eq,k} = 0, \qquad (51)$$

where

$$-N \leq k \leq N, \qquad 1 \leq eq \leq number\ of\ equations$$

are subsequently resolved in the frequency domain.

Multifrequential ansatz

The numerical treatment presented by Rauschenberger [13] is similar to the one of Hall et al. shown in the previous section. Eqn. (49) is extended with a virtual time τ

$$V\frac{\partial Q^*}{\partial \tau} + R(Q)^* + ViAK_{multi}A^{-1}Q^* = 0, \qquad (52)$$

in order to gain a steady solution of the physical instants by iterating the solution of eqn. (52).

4.4 Boundary conditions

In order to avoid the high computational cost, turbomachinery simulations are often not carried out on the whole circumference of the annulus. At a stable operating point, the flow can be considered periodic in the circumferential direction and thus, only a periodic fraction of the wheel can be computed.

Gopinath et al. [15] presented a modified periodic boundary condition for the Harmonic Balance Technique

$$\hat{Q}_k(x, r, \omega + \omega_G) = \hat{Q}_k(x, r, \omega)e^{-iN_k\omega_G} \qquad (53)$$

similar to the phase-lagged condition with the blade shift in circumferential direction ω_G and the nodal distance N_k. Eqn. (53) is applied to the lower and upper circumferential boundaries.

5 Summary

In this term paper, the Harmonic Balance method for modeling unsteady nonlinear flows in turbomachinery has been presented. At first, the governing equations as well as the non-linear time domain analysis and the time-linearized frequency-domain analyses were introduced.

In the time domain analysis, the fluid equations are discretized on a computational grid and then marched in time, which is easy to implement and models linear as well as non-linear disturbances. As this approach has to be both time accurate and stable, the maximum step size is limited, which eventually leads to large computational times.

Using a frequency domain approach, at first the steady flow is computed and then, assuming harmonic unsteadiness in the flow, the fluid equations are transferred to the frequency domain. The resulting equations are easy and computational inexpensively to solve but cannot model dynamic non-linearities and only one frequency per computational session can be taken into consideration.

After giving a brief introduction to the state of the art, the Harmonic Balance Technique is presented. By making use of the periodicity of the turbomachinery, the set of conservation variables can be represented by Fourier coefficients and the periodic URANS computation is casted in several coupled steady flow computations. The HBT can be further divided into the Monofrequential Harmonic Balance Technique, where one frequency and its harmonics are solved and the Multifrequential Harmonic Balance Technique, where multiple base frequencies and their harmonics are solved. The last part is eventually the numerical treatment and the boundary conditions.

The resulting computational method is very efficient and thus at least one or two orders of magnitude faster than the common CFD scheme presented in chapter 3.1 [9].

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