Chair for Computational Analysis of Technical Systems
Faculty of Mechanical Engineering, RWTH Aachen University

Isogeometric Collocation Method

Seminararbeit

By

Marko Blatzheim

Supervisors: Dr. Stefanie Elgeti

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1 Introduction

The exact geometry can not be represented by most popular simulation techniques such as the finite element method or finite differences. Isogeometric analysis employs the functions describing the geometry directly for calculations instead of converting it to a simpler grid first. To create such a geometry Computer-Aided Design (CAD) software is common, which is predominantly based on Non-Uniform Rational B-Splines (NURBS). Among the implementations of isogeometric analysis, Galerkin approaches are the most common. For these methods numerical quadrature rules are the bottleneck because they are optimal for polynomials, which are the basis functions in standard finite element methods, but sub-optimal for isogeometric basis functions. In this seminar work, a different numerical scheme is described: The isogeometric collocation method. To find advantages and disadvantages compared to the isogeometric Galerkin method, the isogeometric collocation method has been implemented in Matlab. In chapter 2 a brief mathematical description of B-splines and NURBS is given followed by an explanation of the basic idea of collocation methods. In chapter 3 the result of a 2D testcase example is shown and analyzed and in chapter 4 conclusions are summarized.

2 Methods

2.1 B-Splines and NURBS

Non-Uniform Rational B-Splines (NURBS) are common in CAD systems. In the following, B-splines and NURBS are described. B-splines are parametric geometry representations based on a linear combination of B-spline basis functions N,

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{B}_i, \tag{2.1}$$

where the coefficients $\mathbf{B}_i$ are points in physical space and referred to as control points. To define the basis functions, the so called knot vector needs to be introduced. A knot vector U is a set of monotonically increasing real numbers in the parametric space,

$$U = \{\xi_1 = 0, ..., \xi_{n+p+1} = 1\} \tag{2.2}$$

where n represents the number of basis functions and p is the order of the B-spline. B-spline basis functions are defined recursively. For $p = 0$ the basis functions are piecewise constant functions as

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

and for $p \geq 1$

$$N_{i,j}(\xi) = \begin{cases} \frac{\xi-\xi_i}{\xi_{i+p}-\xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}} N_{i+1,p-1}(\xi) & \text{if } \xi_i \leq \xi < \xi_{i+p+1} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The continuity of the B-spline is C^{p-1} if internal knots are not repeated. Multi-dimensional B-splines can be constructed by tensor products. In 2D this leads to the surface representation as

$$S(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)\mathbf{B}_{i,j}, \tag{2.5}$$

where $N_{i,p}$ and $M_{j,q}$ are one dimensional basis functions of order p and q, respectively. The derivative of a B-spline is again a B-spline and the corresponding basis functions can be calculated directly from the knot vector. B-splines cannot represent conic sections such as circles or ellipses exactly, but NURBS are able to represent such geometries. Given a B-spline basis function, the NURBS basis function can be expressed as

$$R_i^p(\xi) = \frac{N_{i,p}(\xi)\omega_i}{\sum_{k=1}^{n} N_{k,p}(\xi)\omega_k}, \tag{2.6}$$

with ith weight ω_i. The NURBS geometry is defined by

$$C(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\mathbf{B}_i = \frac{\sum_{i=1}^{n} N_{i,p}(\xi)\,\omega_i\,\mathbf{B}_i}{\sum_{k=1}^{n} N_{k,p}(\xi)\,\omega_k} = \frac{A(\xi)}{W(\xi)}, \tag{2.7}$$

and can be extended to multi dimensional space analogously to B-splines. The NURBS derivative can be calculated by the chain rule and product rule as

$$C'(\xi) = \frac{A'(\xi) - W'(\xi)C(\xi)}{W(\xi)}. \tag{2.8}$$

2.2 Collocation Method

The Collocation method is a numerical scheme to solve differential equations. The general boundary-value problem

$$\begin{cases} \mathcal{D}u = f & \text{in } \Omega, \\ \mathcal{G}u = g & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

with differential operators $\mathcal{D}$ and $\mathcal{G}$ is used to explain the method (Auricchio et al., 2010). A set of collocation points $x_i \in \mathcal{X}$ is used to calculate the approximate solution u^h such that

$$\begin{cases} \mathcal{D}u^h(x_i) = f(x_i) & \text{for } i = 1...m, \\ \mathcal{G}u^h(x_j) = g(x_j) & \text{for } j = 1...l, \end{cases} \tag{2.10}$$

is satisfied, where m is the number of collocation points and l is the number of boundary collocation points. The stability and good behaviour highly depends on the choice of collocation points. In the isogeometric collocation example the so called Greville abscissae of the knot vector was used to generate the collocation points as

$$\bar{\xi}_i = \frac{\xi_{i+1} + \xi i + 2 + ... + \xi i + p}{p}. \tag{2.11}$$

The specific problem to be solved is

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

This means the second derivatives of the NURBS are necessary. Using A and W as introduced in (2.7) the second derivatives of the basis functions can be calculated as

$$\frac{\partial^2 S}{\partial \xi^2} = \frac{1}{W} \cdot \frac{\partial^2 A}{\partial \xi^2} - \frac{2}{W^2} \cdot \frac{\partial A}{\partial \xi} \cdot \frac{\partial W}{\partial \xi} + A \cdot \frac{2}{W^3} \cdot \left(\frac{\partial W}{\partial \xi}\right)^2 - \frac{A}{W^2} \cdot \frac{\partial^2 W}{\partial \xi^2}, \tag{2.13}$$

$$\frac{\partial^2 S}{\partial \eta^2} = \frac{1}{W} \cdot \frac{\partial^2 A}{\partial \eta^2} - \frac{2}{W^2} \cdot \frac{\partial A}{\partial \eta} \cdot \frac{\partial W}{\partial \eta} + A \cdot \frac{2}{W^3} \cdot \left(\frac{\partial W}{\partial \eta}\right)^2 - \frac{A}{W^2} \cdot \frac{\partial^2 W}{\partial \eta^2}, \tag{2.14}$$

$$\begin{aligned} \frac{\partial^2 S}{\partial \xi \partial \eta} &= \frac{1}{W} \cdot \frac{\partial^2 A}{\partial \xi \partial \eta} - \frac{1}{W^2} \cdot \left(\frac{\partial A}{\partial \xi} \cdot \frac{\partial W}{\partial \eta} + \frac{\partial A}{\partial \eta} \cdot \frac{\partial W}{\partial \xi}\right) \\ &+ A \cdot \frac{2}{W^3} \cdot \left(\frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \eta}\right) - \frac{A}{W^2} \cdot \frac{\partial^2 W}{\partial \xi \partial \eta} \end{aligned} \tag{2.15}$$

with respect to the parametric coordinates ξ and η. For the equations, derivatives with respect to physical space are needed. The Jacobian J, the Hessian H and the matrix of squared first derivatives Q are introduced as

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \tag{2.16}$$

$$H = \begin{pmatrix} \frac{\partial^2 x}{\partial \xi^2} & \frac{\partial^2 x}{\partial \xi \partial \eta} & \frac{\partial^2 x}{\partial \eta^2} \\ \frac{\partial^2 y}{\partial \xi^2} & \frac{\partial^2 y}{\partial \xi \partial \eta} & \frac{\partial^2 y}{\partial \eta^2} \end{pmatrix}, \tag{2.17}$$

$$Q = \begin{pmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 \\ 2 \cdot \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} & 2 \cdot \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta} \\ \left(\frac{\partial y}{\partial \xi}\right)^2 & \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} & \left(\frac{\partial y}{\partial \eta}\right)^2 \end{pmatrix}, \tag{2.18}$$

to transform the derivatives from parameter space to physical space, with spatial coordinates x and y. According to (Schillinger et al., 2013) the first derivatives of the basis functions with respect to the physical space can be calculated as

$$\begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix} = J^{-T} \cdot \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix} \tag{2.19}$$

and the second derivatives can be calculated as

$$\begin{pmatrix} \frac{\partial^2 N}{\partial x^2} \\ \frac{\partial^2 N}{\partial x \partial y} \\ \frac{\partial^2 N}{\partial y^2} \end{pmatrix} = Q^{-T} \cdot \left(\begin{pmatrix} \frac{\partial^2 N}{\partial \xi^2} \\ \frac{\partial^2 N}{\partial \xi \partial \eta} \\ \frac{\partial^2 N}{\partial \eta^2} \end{pmatrix} - H^T \begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix} \right). \tag{2.20}$$

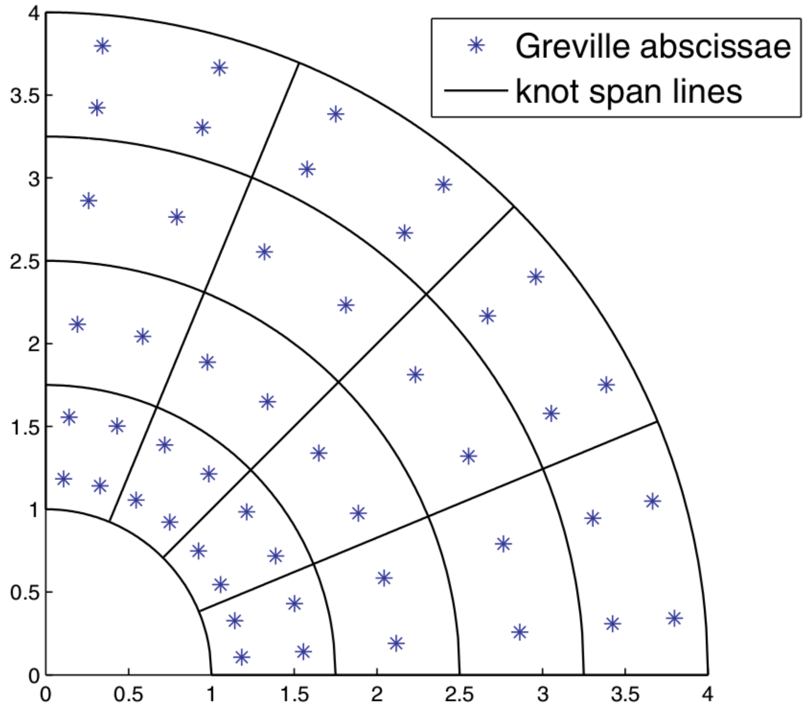

Figure 2.1: Geometrical domain Ω with interior Greville abscissae shown for a quadradic NURBS with 8x10 control points (Auricchio et al., 2010).

The domain Ω for the differential equation is the quarter of an annulus as shown in figure 2.1 with right hand side

$$\begin{aligned} f &= (3x^4 - 67x^2 - 67y^2 + 3y^4 + 6x^2y^2 + 116)\sin(x)\sin(y) \\ &+ (68x - 8x^3 - 8xy^2)\cos(x)\sin(y) \\ &+ (68y - 8y^3 - 8yx^2)\cos(y)\sin(x), \end{aligned} \tag{2.21}$$

such that the exact solution

$$u = (x^2 + y^2 - 1)(x^2 + y^2 - 16)\sin(x)\sin(y) \tag{2.22}$$

is known and can be used for convergence analysis.

3 Results

The isogeometric collocation method leads to a reasonable approximation. Since for equation (2.12) the analytic solution is known, the numerical result can be compared, as seen in Figure 3.1.

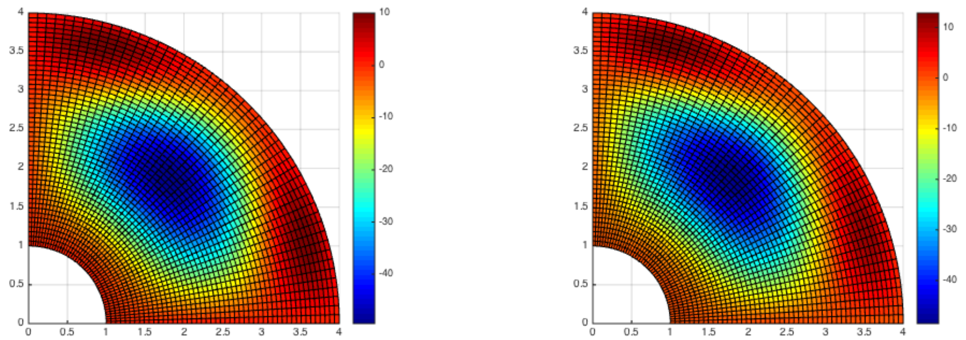

Figure 3.1: The analytic result (left) compared with the isogeometric collocation approximation(right).

The behaviour of the relative error with respect to the refinement can be compared with the theoretical convergence rate. Figure 3.2 shows this for different basis function degree. It can be seen, that the theoretical convergence rate corresponds to the actual convergence.

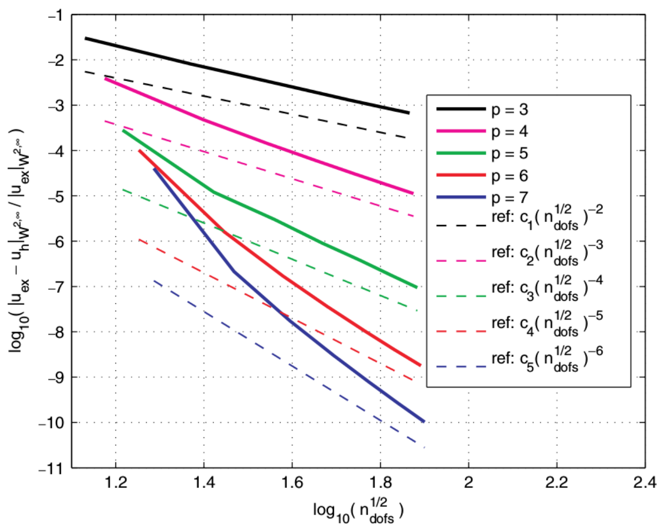

Figure 3.2: The convergence rate for different NURBS degrees (Auricchio et al., 2010).

4 Conclusion

The convergence results show that the isogeometric collocation method is an option to solve differential equations. The method achieves the same convergence rate as isogeometric Galerkin approaches. The bottleneck of Galerkin methods are the choice of quadrature points which are sub-optimal and that the element values needs to be assembled over all quadrature points. For isogeometric collocation methods the choice of collocation points is crucial and currently sub-optimal. As equations (2.13),(2.14) ,(2.15) and (2.20) illustrate, calculating second derivatives of NURBS is costly and thus the efficiency might be the same as with a Galerkin method, where the weak formulation is considered and therefore only first derivatives, but for multiple quadrature points, are calculated. Neither the Matlab code for Galerkin nor for collocation method are optimized so a reliable performance analysis is not possible. Using just B-splines for the solution basis functions leads to improved performance because the second derivative of a B-spline is calculated as quickly as the first derivative, while for NURBS the terms become more complicated for higher order deriatives due to the chain rule and the product rule. A problem with isogeometric collocation may occur if the geometric representation has singularities at the boundary and thus the spline derivatives cannot be transformed from parameter space to physical space. The Greville abscissae always sets collocation points on the boundary while quadrature rules are not considering the values on the boundary. Therefore Galerkin approaches might be less sensitive about singularities in the geometric representation. Well established stabilization techniques developed for Galerkin based methods to solve advection-diffusion or Navier-Stokes problems are not available for collocation methods.

Bibliography

F Auricchio, L Beirao Da Veiga, TJR Hughes, A_ Reali, and G Sangalli. Isogeometric collocation methods. *Mathematical Models and Methods in Applied Sciences*, 20(11): 2075–2107, 2010.

Dominik Schillinger, John A Evans, Alessandro Reali, Michael A Scott, and Thomas JR Hughes. Isogeometric collocation: Cost comparison with galerkin methods and extension to adaptive hierarchical nurbs discretizations. *Computer Methods in Applied Mechanics and Engineering*, 267:170–232, 2013.