# Application of the Isogeometric Concept on the Immersed Boundary Method

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# Content

## 1. Introduction

This term paper was developed during the CES-seminar at RWTH Aachen University. Its aim is to illustrate the ideas of an application of the isogeometric concept to the immersed boundary method that were originally first mentioned in the main reference [2]. To reduce the effort in finite element simulations, the isogeometric design-through-analysis methodology was developed. This concept bases on adaptive hierarchical refinement of B-splines, immersed boundary methods and T-spline computer aided design (CAD) surfaces. The proposed method leads to efficient numerical simulations with highly reduced computational costs.

## 2. Immersed Boundary Method

The core ability of the Immersed Boundary Method is to address particularly complex moving boundary and interface problems. Most of the bodies, which want to be simulated, have a complex boundary structure. The main idea is to choose a domain of simple geometry like a quad which embeds the exact geometry. This domain is linked with a simple Cartesian mesh. To define the boundary of the actual geometry, additional information for example B-splines and NURBS are used. The result is that the whole domain consists of three different types of cells. They are classified as the cells that are outside of the real geometry, the cells that are inside of the body and the cells which are cut by the boundary of the real geometry. The cells outside of the boundary are no longer considered because they do not affect the real geometry. The cells inside of the body are treated as standard finite elements. The cut cells have to be considered with special techniques to treat the boundary accurately. The Immersed Boundary Method has been enhanced to include adaptive, locally refined meshes, but the stable and accurate handling of the cut cells is still a fundamental problem. A possibility to solve this issue is the application of the isogeometric concept to the Immersed Boundary Method.

## 3. Isogeometric Analysis

Isogeometric analysis was introduced by Hughes et al. [1] to close the gap between computer aided design (CAD) and finite element analysis. The main idea behind isogeometric analysis is that the basis used to exactly model the geometry will also serve as the basis for the solution space of the numerical method. This strategy skips the costly mesh generation process which is required for standard finite element analysis and supports a well connected interaction between CAD and finite element analysis tools [3]. This concept has the potential to reduce the time required for the analysis of complex engineering designs by up to 80%. Isogeometric analysis is based on B-splines. Because of the relative simplicity and ubiquity in

CAD tools, advanced techniques like non-uniform rational B-splines (NURBS) and T-splines are also often used. Unlike the nodal representation of geometries in finite element analysis, the NURBS geometries can attain the smoothness of real bodies so that no contact problems occur. The different representations in finite element analysis and isogeometric analysis of two bodies which have sliding contact are illustrated in fig. 1.

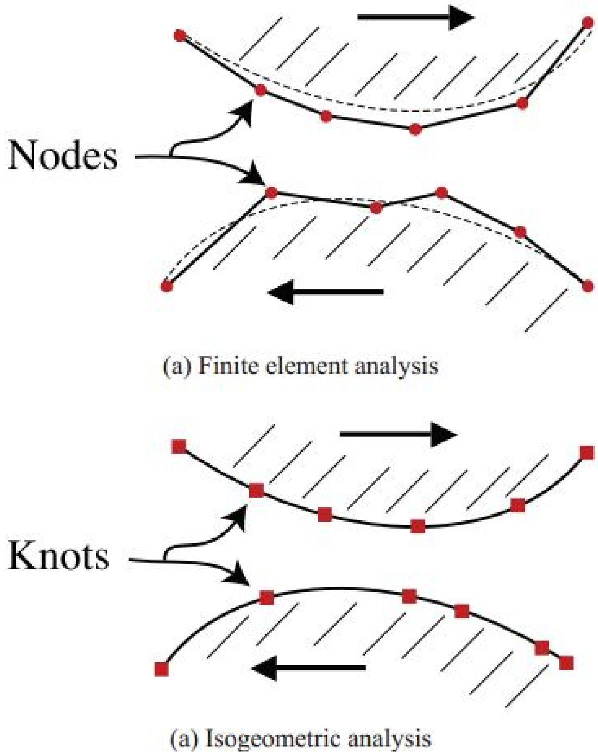



**Figure 1: Attaining smooth representation of bodies with NURBS [1]**

It is obvious that for many practically relevant geometries, Cartesian meshes are not flexible enough and for some geometries, like spherical objects, not possible. Therefore, it is advantageous to describe complex geometries with B-splines or techniques like NURBS and T-splines. In this work the B-spline representation the main focus.

### 3.1 B-Splines

At first the B-spline basis for isogeometric analysis are introduced, for understanding the following concepts and algorithms of their hierarchical refinement. According to [1] ‘Unlike in standard finite element analysis, the B-spline parameter space is local to patches rather than elements. Each element in the physical space is the image of a corresponding element in the parameter space, but the mapping itself is global to the whole patch, rather than to the elements themselves. Patches play the role of sub-domains within which element types and material models are assumed to be uniform. Many simple domains can be represented by a single patch'.

B-spline curves in $\mathbb{R}$ are constructed by taking a linear combination of B-spline basis functions, just as in classical finite element analysis. The vector-valued coefficients of the basis functions are referred to as control points. The B-spline basis of degree $\boldsymbol{p}$ is formed from a knot vector $\boldsymbol{U} = [\boldsymbol{\xi_1}, \boldsymbol{\xi_{n+p+1}}]$ with knots $\boldsymbol{\xi_i}$ and the number of basis functions $\boldsymbol{n}$. Knots may be repeated to influence the smoothness of the B-spline basis. The basis is $\boldsymbol{C^{p-k}}$ at that location, where $\boldsymbol{k}$ is the multiplicity of the knot. In the case of $\boldsymbol{p = k}$, the basis of the B-spline is interpolatory. For open knot vectors, the first and last knot value is repeated $\boldsymbol{p + 1}$ times, which yields to $\boldsymbol{C^{-1}}$ and makes the basis interpolatory. The knot vector is in general a collection of break points and the basis functions are defined by the recurrence formula of Cox de Boor and Mansfield:

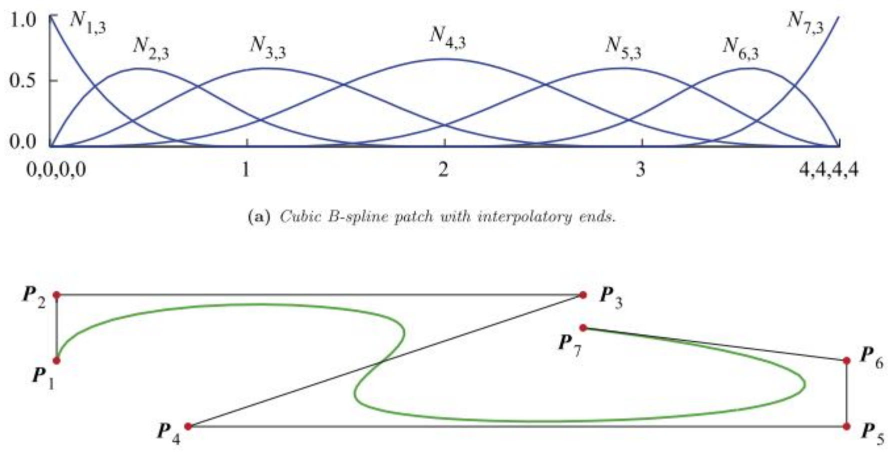
$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & else \end{cases} \qquad \text{for } p = 0$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \qquad \text{for } p \in \mathbb{N}_+$$

The B-spline curve $C(\xi)$ is built by a linear combination of basis functions, where the coefficients $P_i$ are the so called control points.

$$C(\xi) = \sum_{i=1}^{n} \underbrace{N_{i,p}(\xi)}_{\substack{B-Spline \\ basisfunctions}} P_i$$

An example of a B-spline curve of degree $p = 3$ and the knot vector $U = [0,0,0,0,1,2,3,4,4,4,4]$ is shown in fig. 2.




(a) *Cubic B-spline patch with interpolatory ends.*

(b) *B-spline curve generated from the above basis using control points $P_i$.*

**Figure 2: B-spline basis functions and resulting B-spline curve [2]**

For geometry descriptions in multiple dimensions, multivariate B-splines are used. They are a tensor product generalization of the described univariate B-splines and are treated analogously.

### 3.2 Hierarchical refinement of B-splines

Adaptive local refinement algorithms can be used for example to resolve internal and boundary layers in advection dominated flows and stress concentrations in structures [3]. In contrast to the standard nodal basis of finite element analysis, a multivariate B-spline basis does not provide a natural possibility for local mesh refinement. A great advantage of uniform B-splines is their natural refinement by subdivision.

In the following, the B-spline subdivision is explained to show how this concept can be used to set up a hierarchical scheme for local refinement of B-spline basis functions. The scheme is explained on the basis of univariate B-splines. It combines an intuitive principle, full analysis suitability and straightforward implementation.

A B-spline can be expressed as a linear combination of contracted, translated and scaled copies of itself. An example is illustrated in fig. 3 for B-splines of different polynomial degrees. Because of the tensor product structure of multivariate B-splines, the more complex generalization of subdivision is a straightforward extension of their linear combination expression.

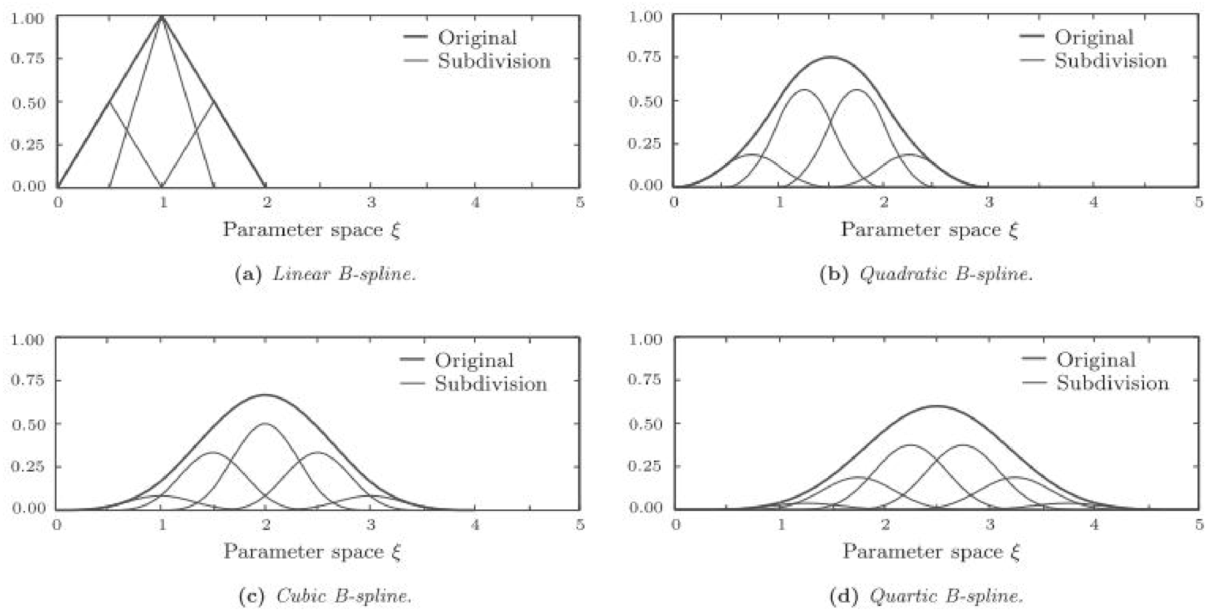



**Figure 3: Subdivision of B-spline basis functions [2]**

The hierarchical scheme for local refinement of B-spline basis functions is illustrated by a short explanation of the following four main refinement steps. To not exceed the frame of this work just the core aspects of the refinement scheme are presented. For a more detailed explanation it is referred to the literature [2].

(1) Two-level hierarchical refinement for one element (Nucleus operation): Adding an overlay of three B-splines of contracted knot span width to the original B-spline basis. Because of the linear independence of single B-splines of contracted knot span width to original B-splines of full knot span width, no changes in the original basis functions are required. The refinement rule introduced in fig. 4 reflects the two-level hierarchy between the original basis and its refinement overlay.

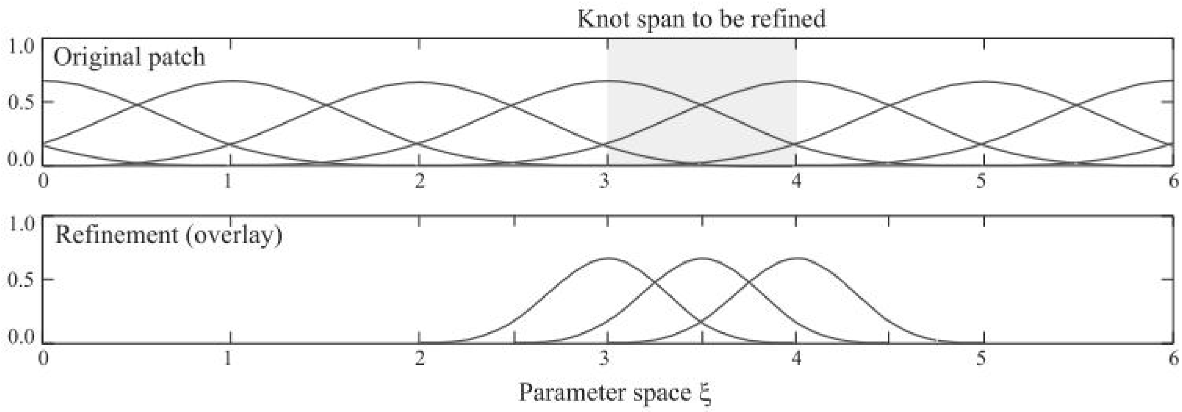



**Figure 4: Refinement of a knot span [2]**

(2) Multi-level hierarchy: Repetition of the procedure described in the last point leads to an increase of the degree of local refinement. With this approach a general multi-level hierarchy, which consists of several overlay levels is established. The multi-level refinement procedure is illustrated in fig. 5, where the nucleus operation is successively applied to the three rightmost knot span elements of each level $k$.

(3) Recovering linear independence: To guarantee full analysis suitability of the hierarchically refined B-spline basis, its linear independence has to be ensured. For this purpose, all B-spline basis functions that are a combination of refined B-spline basis functions of the next level $k+1$ are identified and removed from the hierarchical basis. The basis functions which have to be removed are shown in fig. 5 as dotted lines.

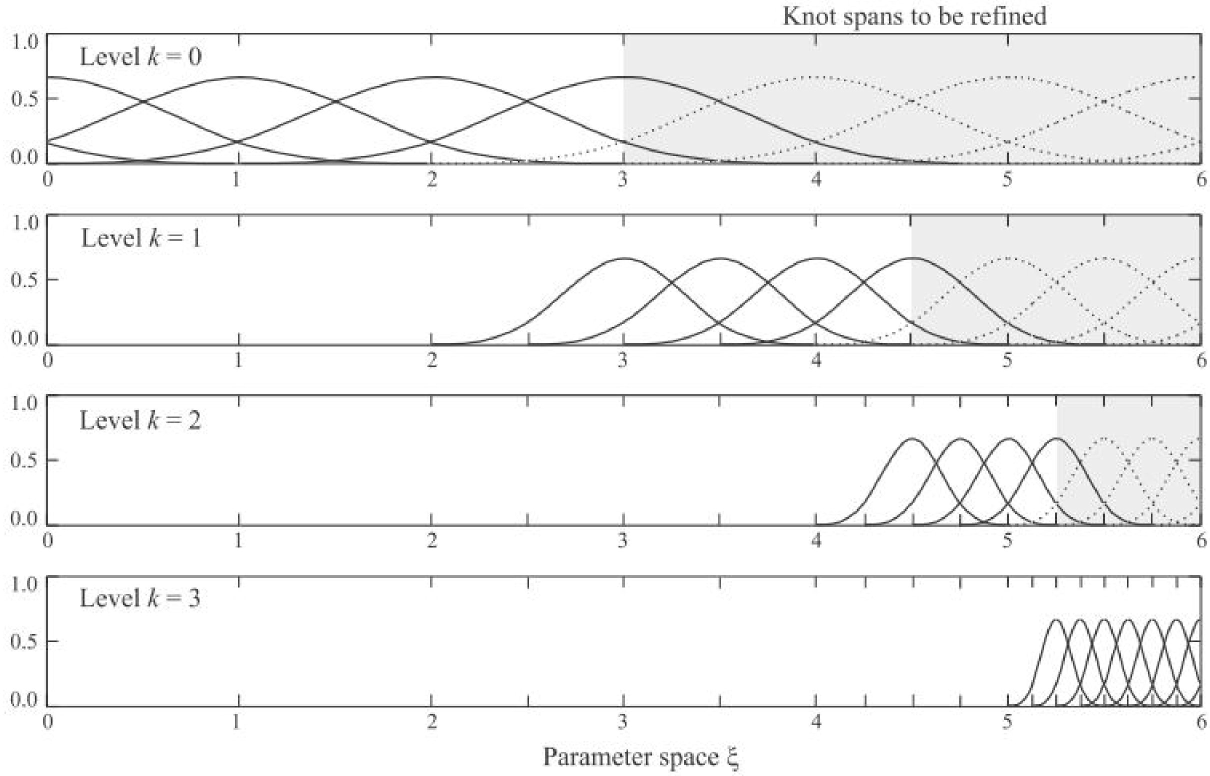



**Figure 5: Hierarchical refinement scheme and removing basis functions (dotted lines) [2]**

(4) Dirichlet constraints: In case of hierarchical refinement, the Dirichlet boundary conditions have to be reconsidered. On one side, they can be incorporated weakly by variational methods or on the other side strongly by a least squares fit of boundary basis functions. The homogeneous boundary conditions can be imposed strongly by removing all basis functions with support at the Dirichlet boundary from the basis.

## 4. Application of the Isogeometric Concept to the Immersed Boundary Method

In the previous sections the basic knowledge of the isogeometric concept and the immersed boundary method was mediated. With this background we opened the door for an application of the Isogeometric concept to the Immersed Boundary Method to reach an isogeometric design-through-analysis methodology to handle the cut cells of the Immersed Boundary Method stably and accurately. The B-spline version of the finite cell method allows a seamless integration of fully three-dimensional parameterizations of complex engineering parts described by T-spline surfaces into finite element analysis. With this procedure we avoid the disadvantages of the translation of complex CAD based geometrical models into finite element discretizations, which is computationally expensive, difficult to automate and often leads to error-prone meshes. Because of the fact that immersed boundary methods don't need body-fitted meshes, the domain is embedded into a regular grid of axis-aligned elements, which can be generated irrespective of the geometric complexity involved. This leads to the possibility of a simple meshing procedure, which can be fully automated.

### 4.1 Finite Cell Method

The Finite Cell Method belongs to the class of immersed boundary methods and it is a strategy for treating boundaries and interfaces. In this work the B-spline version of the finite cell method is considered. It combines the fictitious domain approach with a regular Cartesian grid of axis aligned B-splines, weak imposition of unfitted Dirichlet boundary conditions and an adaptive integration procedure for cut elements. The integration accuracy in cut elements is ensured by adaptive integration sub-cells, which lead to an aggregation of Gauss points around the boundary. The creation of a highly refined quadrature mesh of sub-cells, which surround the boundary and interfaces, is the unique feature of the finite cell method. In the vicinity of boundaries or interfaces, the rectilinear mesh can be locally refined. Quadrature points within the sub-cells are tagged as being either inside the physical domain, or outside of it. The core idea, which consists of the extension of the physical domain of interest outside of its physical boundaries into a larger embedding domain of simple geometry, is illustrated in fig. 6. The whole domain $\Omega$ includes the physical domain $\Omega_{phys}$ and the fictitious domain $\Omega_{fict}$ . This can be meshed easily by a structured grid.

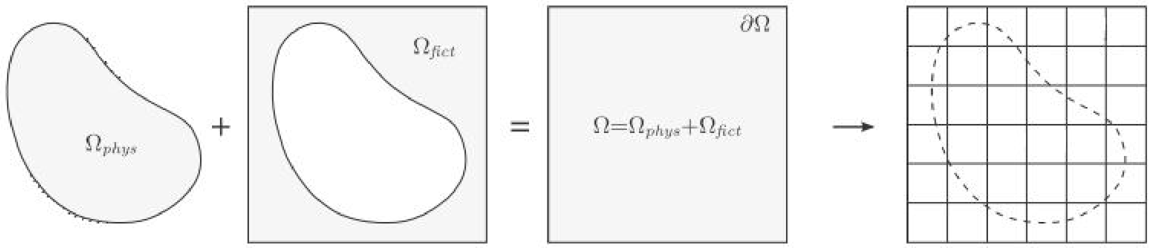



**Figure 6: Embedding the original geometry into a larger domain of simple geometry [2]**

Fig. 7 shows the hierarchical refinement of the structured grid, which is discretized by uniform B-splines, into sub-cells until the desired depth level $k$ is reached. Basis functions without support in the physical domain are eliminated.

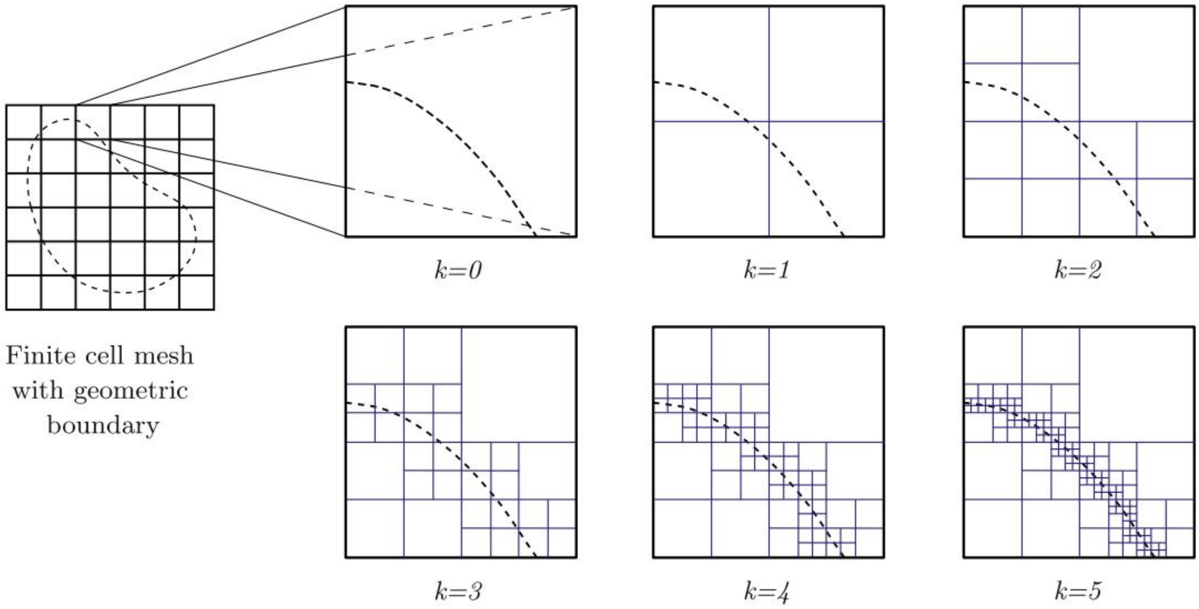



**Figure 7: Hierarchical refinement on structured grid [2]**

The schematic classification of the Cartesian mesh cells as physical or fictitious is shown with the help of a different example. In fig. 8 a rectangular plate with a circular hole is illustrated. Furthermore, each sub-cell contains integration points, which are also sub-divided.

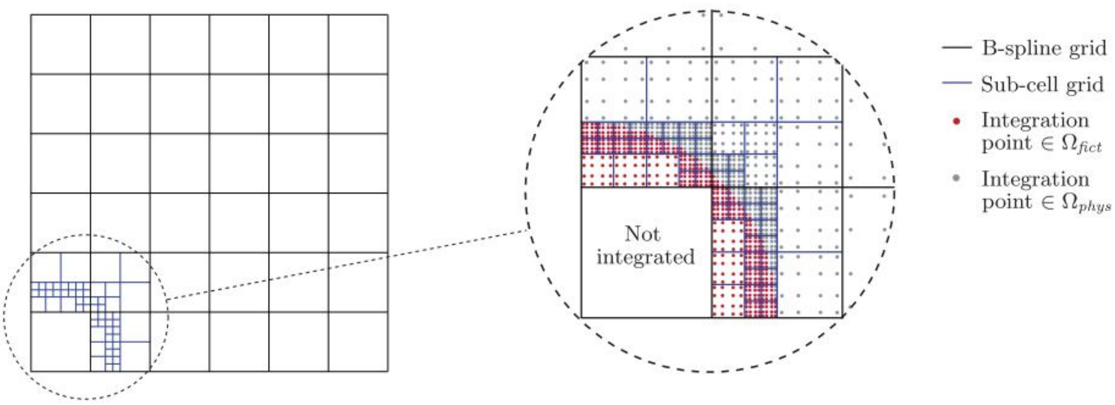



**Figure 8: Classification and sub-division of cells**

The integration accuracy in cut elements is ensured by adaptive integration sub-cells, which lead to an aggregation of Gauss points around the boundary. It is important to note that integration sub-cells do not affect the B-spline basis functions, but only increase the number of integration points around the geometric boundary. The element located completely outside $\Omega_{phys}$ is not integrated. The aggregation of integration points in cut elements ensures that the geometric boundary is taken into account accurately during numerical integration.

## 5. Application

In this section an example is demonstrated to illustrate the application of the isogeometric concept to the immersed boundary method with hierarchical B-spline refinement. The chosen example is a modal analysis of a three dimensional ship propeller. To not exceed the frame of this work, this chapter is focused on the discussed subjects. The application of the ship propeller is treated in [2] more explicitly. The propeller geometry is given by a T-spline surface, see fig. 9. The B-spline version of the finite cell method is applied to demonstrate the discretization procedure in the framework of the immersed boundary concept and its combination with hierarchical refinement. As shown in fig. 10, the complete structure is embedded in a regular grid of axis-aligned B-splines (black lines). In this example uniform B-splines of polynomial degree $p = 3$ are chosen, which offer higher order approximation, but still are computationally efficient due to their relatively local support.

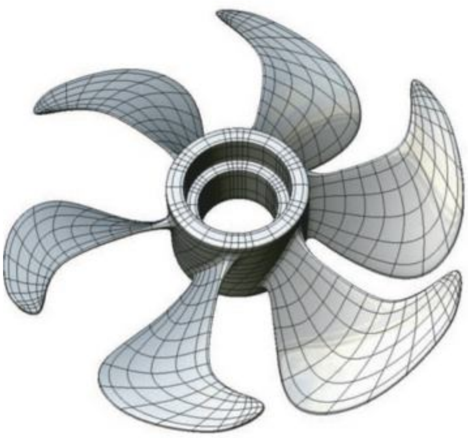


**Figure 9: T-spline surface of a ship propeller [2]**

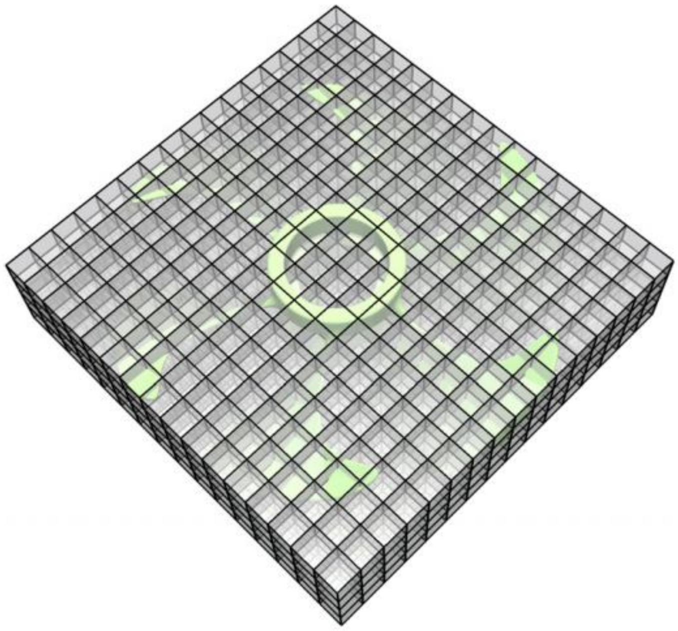


**Figure 10: Propeller is immersed in a bounding box of B-spline elements [2]**

In the following step, all knot span elements without support in the propeller domain $\Omega_{phys}$ are omitted from the discretization, which leads to a reduced set of elements as illustrated in fig. 11. The decision whether an element is to be kept or not is based on a simple point location query, which checks if at least one integration point is located in $\Omega_{phys}$.

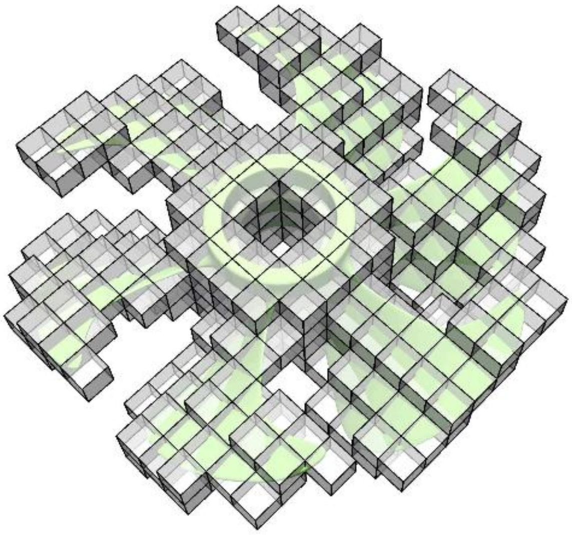


Figure 11: Deletion of cells without support in the propeller domain [2]

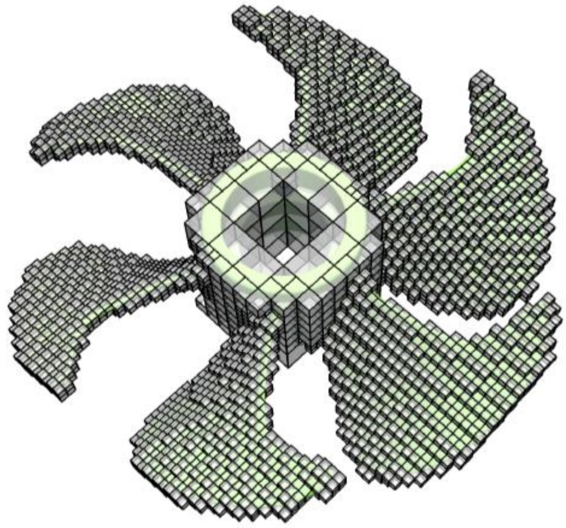


Figure 12: Hierarchical refined mesh [2]

Because of the two different structures of the core propeller and the blades, two levels of hierarchical refinement to the blades are applied, while the discretization of the central hub remains as it is. This yields the adaptive mesh, illustrated in fig. 12, which has only 53.052 degrees of freedom (DOFs), instead of 80.922 DOFs, which would be the result of a uniformly refined mesh. In the next step, each element cut by the geometric boundary is equipped by additional sub-cells, which has been described in more detail in the last section. Like mentioned before, the blue sub-cells of fig. 13 have no influence on the B-spline basis functions, which are still defined over the set of elements shown in black lines in fig. 12.

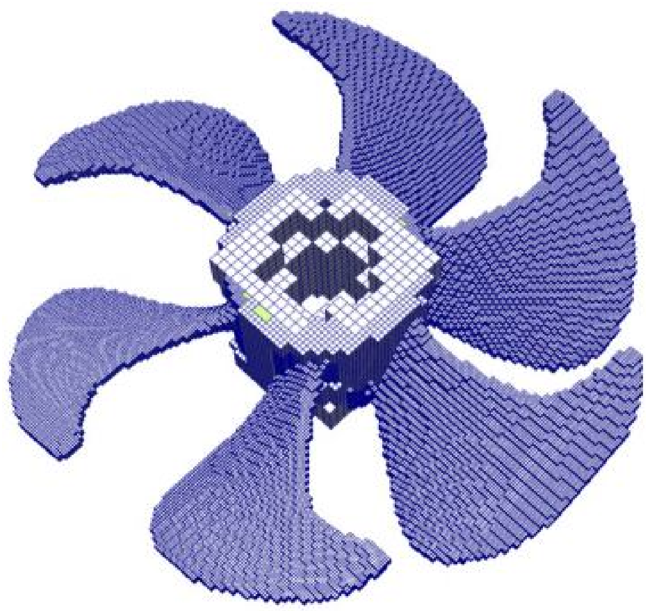


Figure 13: Sub-cell partitioning of the ship propeller (226.272 sub-cells of level k = 2) [2]

In this example each sub-cell contains 4x4x4 Gauss points, leading to an aggregation of integration points in cut elements to accurately take into account the geometric boundary during numerical integration. The result is that the discussed application of the isogeometric analysis to the immersed boundary method, leads to an adaptive mesh, with highly localized refinement.

## 6. Conclusion

In this term paper, the ideas of an application of the isogeometric concept to the immersed boundary method, was illustrated. Also, theoretical implementation aspects for hierarchical refinement of B-splines have been discussed. Referring to the three dimensional application of the ship propeller, the hierarchical refinement as a basis for adaptive B-spline based isogeometric analysis, was successfully tested. The result of the method leads to adaptive meshes with highly localized refinement, and no unnecessary mesh refinement absent from the areas of interest have been experienced. By skipping the expensive mesh generation, manual grid manipulation to gain accurate geometry representations is no longer needed. Therefore, the illustrated method should be established. According to Schillinger [4], there is still more work needed. In the field of analysis, higher requirements of the finite element technology are needed. But finally, the illustrated method is a good way to reduce the effort of numerical simulations with a highly reduced computation costs.

**References**


[1] T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput. Methods Appl. Mech. Engrg. 194 (2005) 4135–4195

[2] D. Schillinger, et al.; An isogeometric design-through-analysis methodology based on adaptive hierarchical refinement of NURBS, immersed boundary methods, and T-spline CAD surfaces, Comput. Methods Appl. Mech. Engrg. (2012)

[3] R.A.K Sanches, P.B. Bornemann, F. Cirak; Immersed b-spline (i-spline) finite element method for geometrically complex domains, Department of Engineering, University of Cambridge (2011)

[4] D.Schillinger; Unsere Methode würde sehr viel teure Arbeitszeit einsparen, http://www.konstruktion.de/themen/hard-und-software/unsere-methode-wurde-.sehr-viel-teure-arbeitszeit-einsparen/, (24.07.2012)