Outflow positivity limiting for hyperbolic conservation laws

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Abstract

Physical solutions to hyperbolic systems of conservation laws typically stay in an invariant domain of positive states. Examples include shallow water, which maintains positivity of the depth, and Euler gas dynamics, which maintains positivity of the density and pressure. Numerical solutions that wander outside the domain of positive states are prone to instability due to loss of hyperbolicity. Finite volume methods (such as WENO or DG) satisfy a discrete conservation law, ensuring that shocks travel at correct speeds, but high-order-accurate methods easily lose positivity of cell averages. To maintain positivity, we cap cell outflow by directly calculating the largest stable time step that maintains positivity of cell averages. This time step can become arbitrarily small, however, halting the simulation. The challenge is therefore to design high-order-accurate numerical fluxes that cap the ratio of the cell outflow to the cell average, thereby guaranteeing a minimum positivity-preserving time step.

We show that the rate of cell outflow is bounded above by twice the product of the boundary average and the maximum wave speed. We use linear damping of the deviation from the cell average to cap the boundary average at the maximum possible for a positive solution in the polynomial representation space of the mesh cell. We cap wave speeds by calculating fluxes with remapped states that enforce physically justified upper bounds on fluid velocity and temperature.

Our framework incorporates and extends the positivity-limiting framework of Zhang and Shu. We allow for mesh cells of arbitrary geometry, we use a cell positivity condition that can be efficiently checked, we allow spatially dependent flux functions, and we ensure the same minimum positivity-preserving time step as if linear damping were sufficient to enforce positivity at every point in the mesh cell. High-order finite volume methods can be outfitted with outflow positivity limiting without loss of order of accuracy and with marginal additional computational expense.