# Numerical solution of hyperbolic moment models for the Boltzmann equation in non-conservative form


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## Moment Models for the Boltzmann Equation

**Aim:** Derivation of hyperbolic PDE systems for the solution of the Boltzmann Equation

$$\frac{\partial}{\partial t} f(t,\boldsymbol{x},\boldsymbol{c}) + c_i \frac{\partial}{\partial x_i} f(t,\boldsymbol{x},\boldsymbol{c}) = S(f)$$

in a $d$-dimensional setting, i.e. position $\boldsymbol{x} \in \mathbb{R}^d$ and velocity $\boldsymbol{c} \in \mathbb{R}^d$.

Firstly, we apply nonlinear transformation of the velocity variable to obtain a Lagrangian velocity phase space and exhibit **physical adaptivity** allowing for efficient discretizations:

$$\boldsymbol{\xi}(t,\boldsymbol{x},\boldsymbol{c}) := \frac{\boldsymbol{c} - \boldsymbol{v}(t,\boldsymbol{x})}{\sqrt{\theta(t,\boldsymbol{x})}}.$$

We expand the distribution function in a series around local equilibrium

$$f(t,\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=0}^{M} f_i(t,\boldsymbol{x}) \mathcal{H}_i(\boldsymbol{\xi})$$

using weighted Hermite polynomials as basis functions.

## Hyperbolicity of a PDE System

The PDE system of the form

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{A}(\boldsymbol{u})\frac{\partial}{\partial x}\boldsymbol{u} = 0$$

is **globally hyperbolic** if the matrix $\boldsymbol{A}$ is diagonalizable with real eigenvalues for every $\boldsymbol{u}$.

Hyperbolicity is necessary for

- Physical solutions with bounded propagation speeds,
- Well-posedness and stability of the solution.

## Hyperbolic Moment Models

Standard projection methods like Grad [2] do not lead to hyperbolic PDE systems. However, new methods overcome this problem and result in globally hyperbolic systems, e.g.

- **Hyperbolic Moment Equations** (HME) by Cai et al. [3],
- **Quadrature-Based Moment Equations** (QBME) by Koellermeier et al. [1].

The resulting system of equations for QBME exhibits various desirable properties:

- Global hyperbolicity,
- Exactness of the first $M-1$ equations,
- Rotational invariance even for the $d$-dimensional case.

## Comparison of the Equation Systems

The model equations can be written as

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}\partial_x \boldsymbol{u}_M = 0,$$

with $\boldsymbol{A}$ depending on the model. For 1D and $M=4$ the different models result in:

**Grad**

$$\begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix}$$

**HME**

$$\begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & v \end{pmatrix}$$

**QBME**

$$\begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2}-\frac{10f_4}{\theta} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta+\frac{15f_4}{\rho\theta} & v \end{pmatrix}$$

Grad's method is only locally hyperbolic around equilibrium.

In contrast to Grad's model it can be shown that with the small changes in the system matrix for HME and QBME the system becomes **globally hyperbolic**.

## Non-conservative Numerical Schemes

The PDE system in non-conservative form requires special numerical schemes for the non-conservative product. Possible candidates are:

- **PRICE-C** scheme by Canestrelli [4]. A first order centered scheme using no information about eigenvalues, extendable to higher order. Reduces to the conservative FORCE scheme if applied to a conservative equation. Applicable to 2D, unstructured grids.
- **Wave Propagation** scheme by LeVeque [5]. A second order upwind type scheme using the eigenvector decomposition of the system matrix and applying limiters to ensure stability of the method. Applicable to 2D, uniform cartesian grids only.

## Shock Tube Test Case

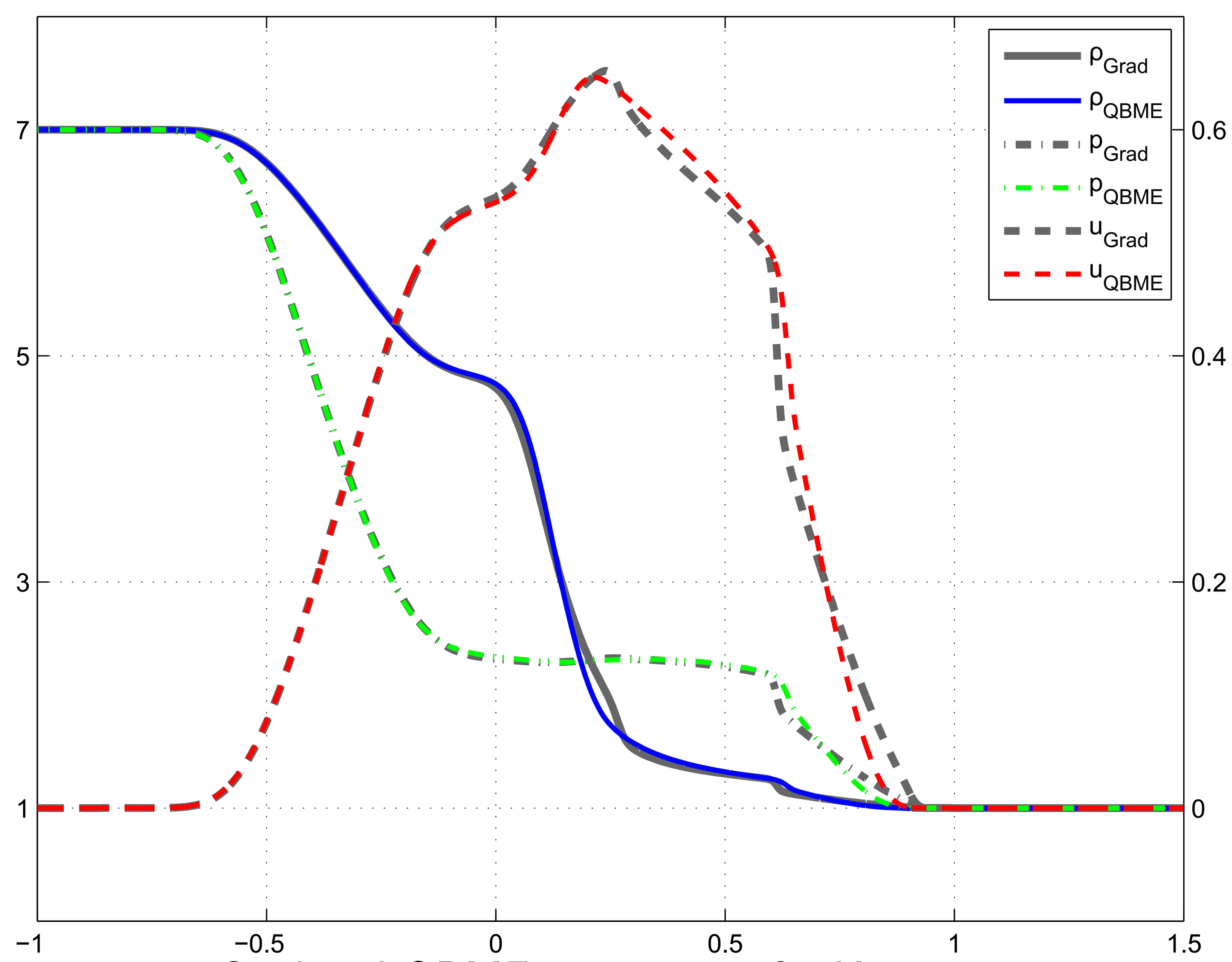
As a test case we use a Riemann problem with BGK collision operator

$$\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} = -\frac{1}{\tau}\boldsymbol{P}\boldsymbol{u},$$

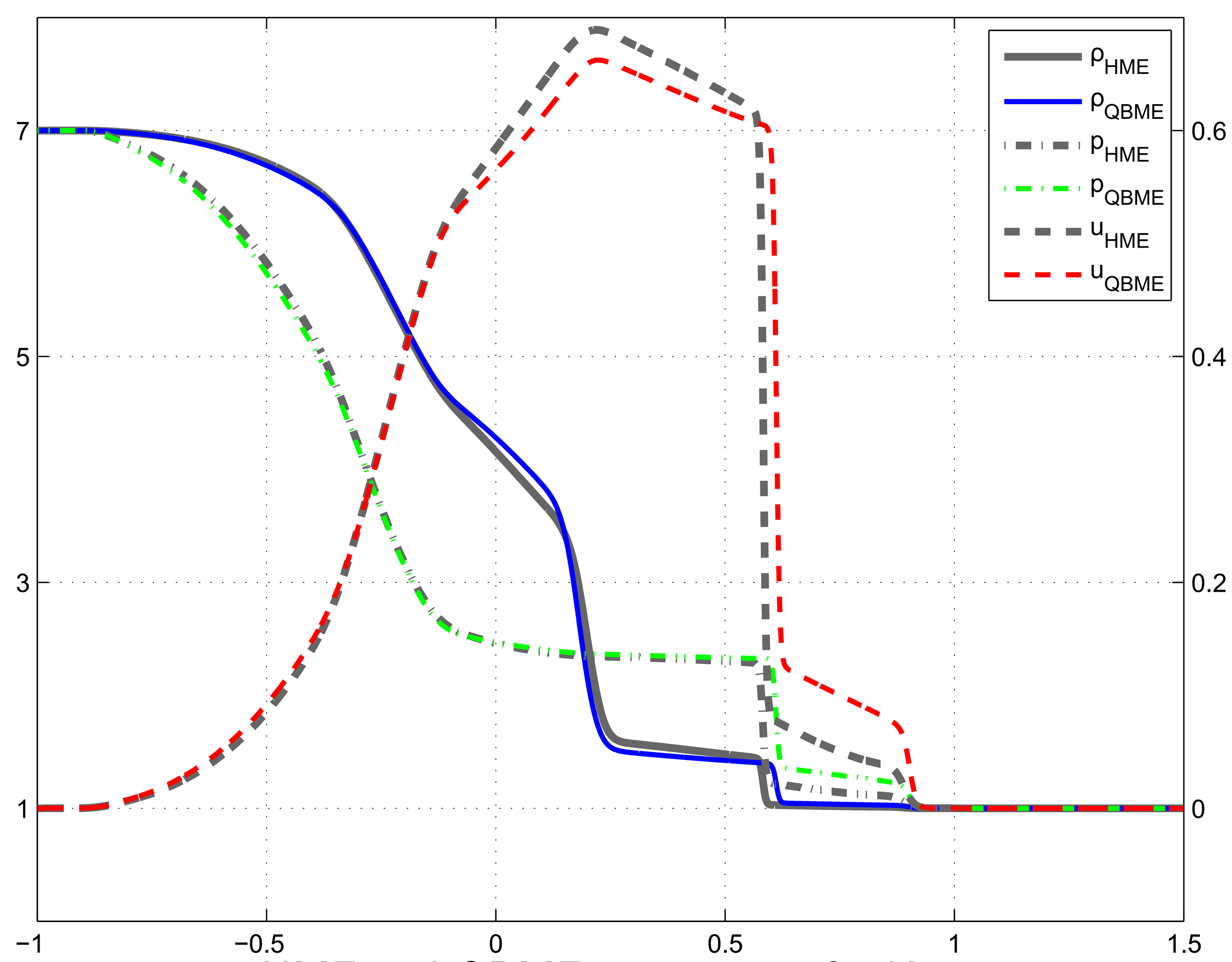
$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_L = (7,0,1,0,0)^T & \text{if } x<0, \\ \boldsymbol{u}_R = (1,0,1,0,0)^T & \text{if } x>0, \end{cases}$$

for $M=4$, variable vector $\boldsymbol{u} = (\rho, u, \theta, f_3, f_4)$ and non-linear relaxation time $\tau = \frac{\mathrm{Kn}}{\rho}$.

## Shock Tube Results




*Grad and QBME comparison for Kn* $= 0.05$.




*HME and QBME comparison for Kn* $= 0.5$.

## References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[2] H. Grad. On the kinetic theory of rarefied gases, *Comm. Pure Appl. Math.*, **2** (1949), 331–407.

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