Extension of standard fluid dynamics equations: Simulation of rarefied gases using hyperbolic moment models

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Motivation

Aim: Extension of standard fluid dynamics for rarefied flows using hyperbolic PDEs:

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{A}(\boldsymbol{u})\frac{\partial}{\partial x}\boldsymbol{u} = \boldsymbol{S}(\boldsymbol{u}),$$

where $\boldsymbol{u} \in \mathbb{R}^M$ is an extended vector of flow variables.
Application: Hypersonic flows in rarefied gases, e.g. high altitude re-entry flights.

Rarefied gases

Rarefied gases are characterized by a large *Knudsen number*

$$\mathrm{Kn} = \frac{\lambda}{L}$$

for mean free path length λ and reference length L.
Large Knudsen numbers occur for

- large λ, e.g. high altitude flows, re-entry
- small L, e.g. channel flows, micro systems (MEMS)

Euler equations

For dense gases the isentropic Euler equations yield a hyperbolic system of PDEs for density ρ, velocity u and temperature $\theta = \frac{p}{\rho}$.

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix} + \begin{pmatrix} u & \rho & 0\\ \frac{\theta}{\rho} & u & 1\\ 0 & 2\theta & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}. \quad (1)$$

The Euler equations (1) are only valid for very small Knudsen number $\mathrm{Kn} \ll 10^{-2}$.

Hyperbolic moment equations

Augmenting the Euler system using additional variables (so-called moments), hyperbolic moment equations (HME) can be derived, e.g. for two additional variables:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ u\\ \theta\\ f_3\\ f_4\end{pmatrix} + \begin{pmatrix} u & \rho & 0 & 0 & 0\\ \frac{\theta}{\rho} & u & 1 & 0 & 0\\ 0 & 2\theta & u & \frac{6}{\rho} & 0\\ 0 & 4f_3 & \frac{\rho\theta}{2} & u & 4\\ -\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ u\\ \theta\\ f_3\\ f_4\end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix}0\\ 0\\ 0\\ f_3\\ f_4\end{pmatrix}, \quad (2)$$

with right-hand side relaxing the higher order variables towards equilibrium $f_3, f_4 = 0$ with relaxation time τ.

Hyperbolic moment models

Additional modifications (marked in red) for hyperbolicity are necessary as standard moment equations are not globally hyperbolic, which causes break down of the numerical solution. Newly derived hyperbolic moment models include

- **Hyperbolic Moment Equations** (HME) by Cai et al. [4],
- **Quadrature-Based Moment Equations** (QBME) by Koellermeier et al. [1],
- **Simplified Hyperbolic Moment Equations** (SHME) by Koellermeier et al. [3]

and an additional model reduction framework allows derivation of more models [5].
The resulting systems of equations exhibit desirable properties:

- Global hyperbolicity,
- Convergence to Euler solution for $\mathrm{Kn} \to 0$
- Rotational invariance for the multi-dimensional case.

Non-conservative numerical schemes

The hyperbolic moment equations (2) are only given in non-conservative form and require special numerical schemes for the non-conservative product. We implement the **PRICE-C** scheme by Canestrelli et al. featuring

- first order centered scheme
- no eigenstructure information necessary
- implemented on 2D unstructured quad grids
- extension to second order

Shock tube test case

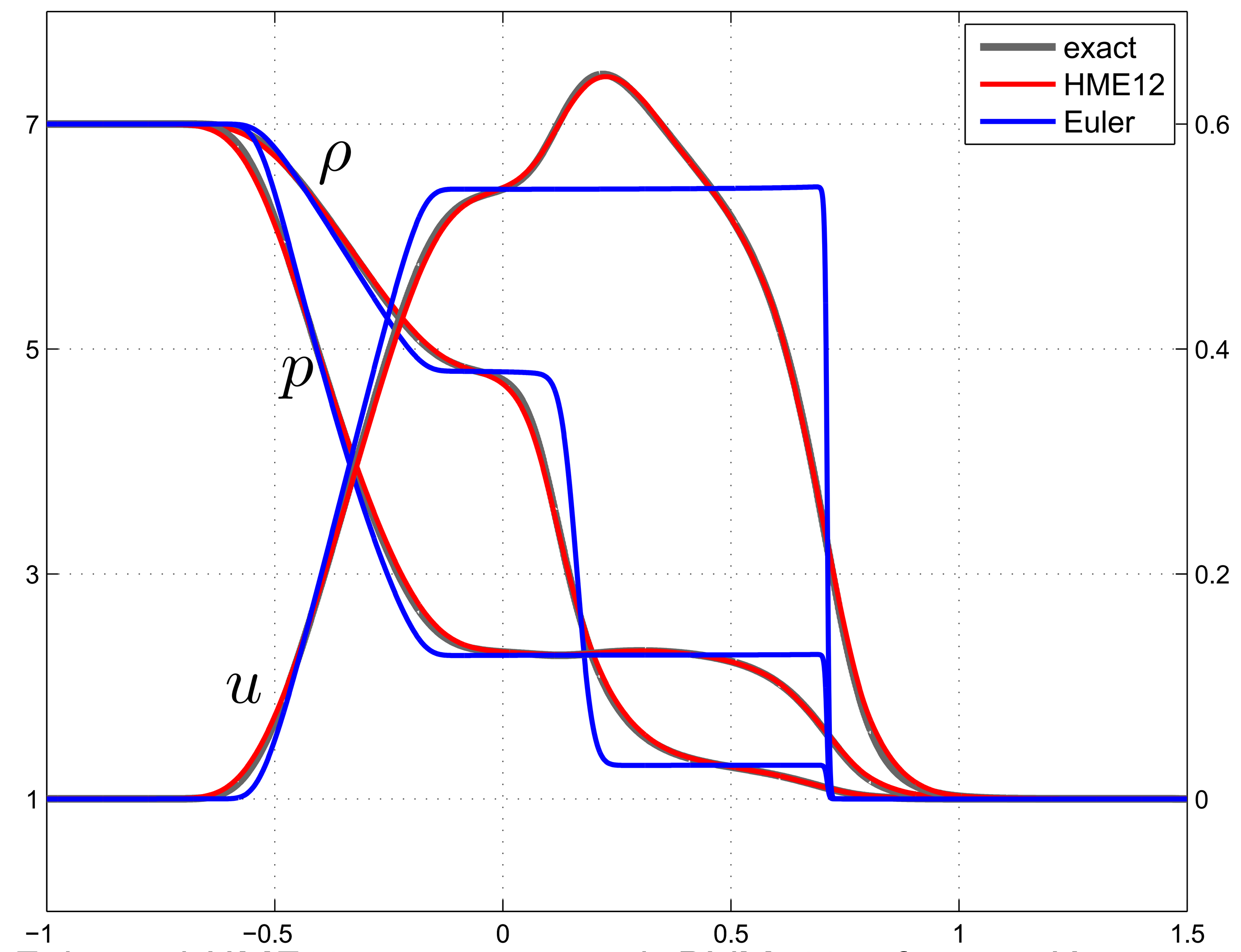
As 1D test case we use a Riemann problem with BGK collision operator [2]

$$\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} = -\frac{1}{\tau}\boldsymbol{P}\boldsymbol{u},$$

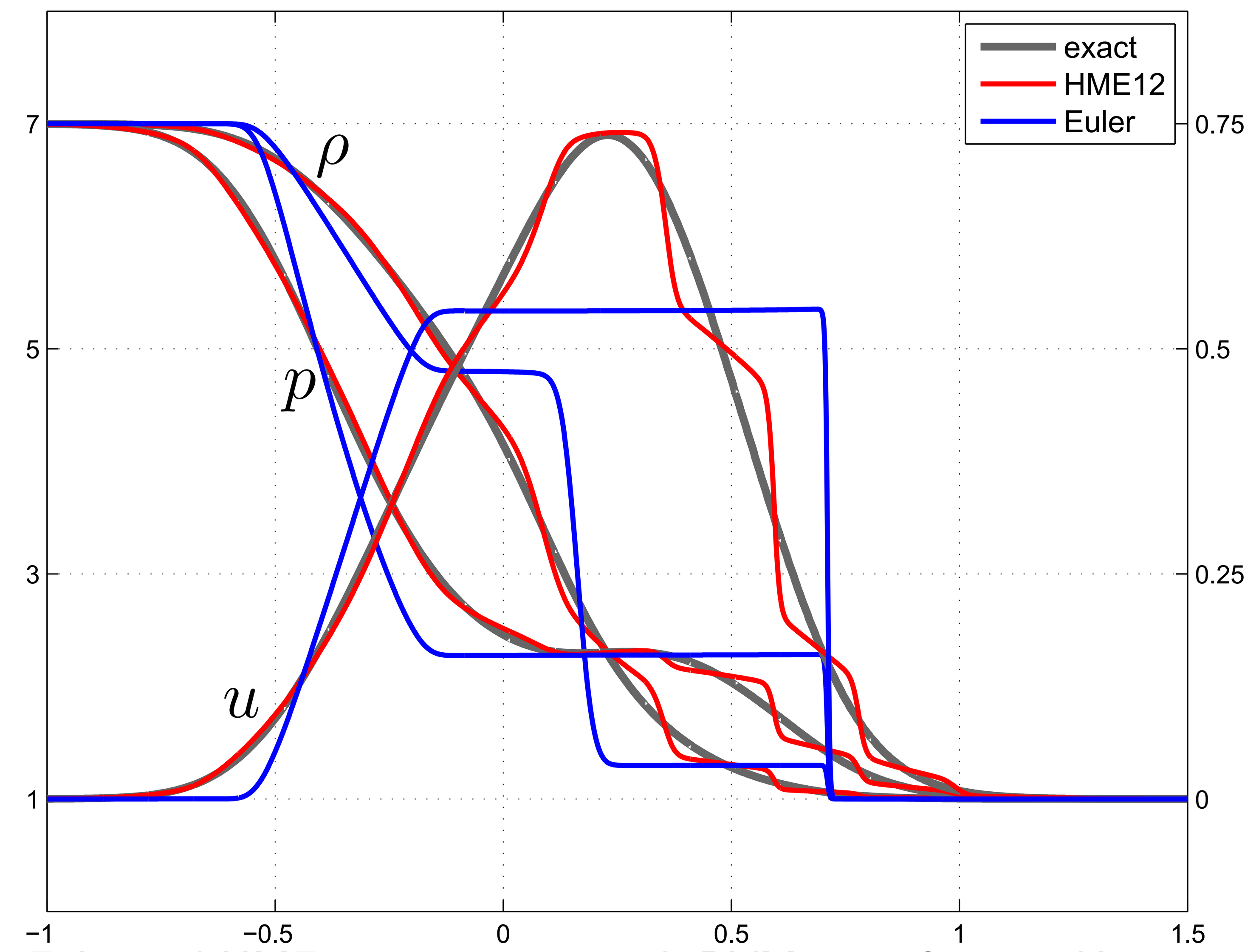
$$\boldsymbol{u}(0,x) = \begin{cases}\boldsymbol{u}_L = (7,0,1,0,\ldots,0)^T & \text{if } x<0,\\ \boldsymbol{u}_R = (1,0,1,0,\ldots,0)^T & \text{if } x>0,\end{cases}$$

for 12 variables $\boldsymbol{u} = (\rho, u, \theta, f_3, \ldots, f_{11})^T$ and non-linear relaxation time $\tau = \frac{\mathrm{Kn}}{\rho}$.

Shock tube results



Euler and HME12 comparison with DVM400 reference, Kn $= 0.05$.



Euler and HME12 comparison with DVM400 reference, Kn $= 0.5$.

References

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[3] J. Koellermeier and M. Torrilhon. Simplified Hyperbolic Moment Equations, submitted to *Springer Proceedings in Mathematics and Statistics*

[4] Z. Cai, Y. Fan and R. Li. Globally hyperbolic regularization of Grad's moment system, *Comm. Pure Appl. Math.*, **67** (2014), 464–518.

[5] Y. Fan, J. Koellermeier, J. Li, R. Li, M. Torrilhon. Model Reduction of Kinetic Equations by Operator Projection, *J. Stat. Phys.*, **162(2)** (2016), 457–486.