Using Quadrature-Based Projection Methods

On the Numerical Solution of the Boltzmann Equation

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Velocity Discretization of the Boltzmann Equation

We consider hyperbolic moment models for the solution of the Boltzmann Equation
\[ \frac{\partial}{\partial t} f(t, x, c) + v \frac{\partial}{\partial x} f(t, x, c) = S(f), \]
where we assume a d-dimensional setting, i.e., we have position \( x \in \mathbb{R}^d \) and velocity \( c \in \mathbb{R}^d \).
We apply a nonlinear transformation of the velocity variable in order to obtain a Lagrangian velocity phase space and exhibit physical adaptivity, which allows for efficient and yet simple discretizations:
\[ \xi(t, x, c) = \frac{c - \mathbb{E}(c|x)}{\sqrt{\mathbb{V}(c|x)}}. \]

Hyperbolic Moment Models

Standard projection methods like Grad [3] do not lead to hyperbolic PDE systems. Recently, different methods have been developed to derive globally hyperbolic systems:
- Hyperbolic Moment Equations (HME) by Cai et al. [4].
- Quadrature-Based Moment Equations (QBME) by Koellermeier et al. [1].

Operator Projection Framework

The discretization in the transformed velocity space leads to an infinite PDE system of the following form
\[ MD \partial_t u + CMD \partial_x u = 0, \]
for an unknown infinite vector \( u = (\rho, v, \theta, f_3, f_4, \ldots) \) and different matrices corresponding to different steps during the derivation of the equation. In order to get a finite set of equations we apply projection operators (see [2] for details).

Properties of the Model Equations

The resulting system of equations for QBME exhibits various desirable properties:
- Global hyperbolicity.
- Exactness of the first \( M - 1 \) equations.
- Rotational invariance even for the \( d \)-dimensional case.

Comparison of the Equation Systems

We write the different models in the following form to allow for comparison:
\[ \partial_t u_M + A_D u_M = 0, \]
where the system matrix \( A \) depends on the model. For the 1D 5-moment case \( M = 4 \) the different models result in the following system matrices:

- Grad (3)
- HME
- QBME

It can be shown that with the small changes in the system matrix for HME and QBME, the system becomes globally hyperbolic.

Next Steps

- Working with new projection operators to obtain more advanced model equations.
- Extension of a numerical solver for the non-conservative QBME and investigation of approximation properties compared to Grad and HME.

References

2. Y. Fan, J. Koellermeier, J. Li, R. Li and M. Torrilhon. Model Reduction of Kinetic Equations by Operator Projection, submitted