On the Stability of Hyperbolic Moment Equations

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Outline

1. Introduction to Moment Methods
2. Stability of Hyperbolic Moment Equations
3. Simulation Results
Introduction to Moment Methods
Boltzmann Transport Equation

\[ \frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f) \]

PDE for particles' probability density function \( f(t, x, c) \)
- Describes change of \( f \) due to transport and collisions
- Collision operator \( S \)
- Usually a 7-dimensional phase space
Model Order Reduction

\[ \frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f) \]

Ansatz in velocity space

\[ f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) H_i \left( \frac{c - v}{\sqrt{\theta}} \right) \]
Model Order Reduction

\[
\frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f)
\]

Ansatz in velocity space

\[
f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) H_i \left( \frac{c - v}{\sqrt{\theta}} \right)
\]

- shifted and scaled weighted Hermite polynomial \( H_i \)
- Galerkin approach leads to finite system of PDEs for coefficients \( f_i \)
A Short History of Hyperbolic Moment Equations

Grad’s Method [Grad, 1949]
- Galerkin projection with Hermite polynomials, locally hyperbolic

Hyperbolic Moment Equations (HME) [Fan et al., 2012]
- modification of system matrix

Quadrature-Based Moment Equations (QBME) [JK, 2013]
- use of Gaussian quadrature

Operator Projection framework (OP) [Fan, JK et al., 2014]
- use of projections
Introduction to Moment Methods

Stability of Hyperbolic Moment Equations

Simulation Results

Boltzmann Equation

Moment Method

Moment System

Boltzmann equation

\[
\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0
\]

\[
\downarrow
\]

Hyperbolic Moment equations

\[
D \frac{\partial}{\partial t} w + MD \frac{\partial}{\partial x} w = 0
\]
Boltzmann equation

\[ \frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0 \]

\[ \downarrow \]

Hyperbolic Moment equations

\[ \frac{\partial}{\partial t} w + D^{-1} MD \frac{\partial}{\partial x} w = 0 \]
Stability of Hyperbolic Moment Equations
Stability of Hyperbolic System

\[ \partial_t u + A \partial_x u = 0 \]

**Wave ansatz**

\[ u = u_0 \cdot e^{i(kx - \omega t)}, \quad k \in \mathbb{R}, \omega \in \mathbb{C}, \quad \text{Im}(\omega) \leq 0 \text{ for stability.} \]

**Stability Analysis**

\[ -i\omega u + ikAu = 0 \]
\[ (kA - \omega I)u = 0 \]

\[ \omega = EV(kA) = k \cdot EV(A) \]

⇒ A needs to have only real eigenvalues.
Stability of Relaxation System

\[ \partial_t u = \varepsilon B u \]

Wave ansatz

\[ u = u_0 \cdot e^{i(kx - \omega t)}, \quad k \in \mathbb{R}, \omega \in \mathbb{C}, \quad \text{Im}(\omega) \leq 0 \text{ for stability.} \]

Stability Analysis

\[ -i\omega u = \varepsilon Bu \]
\[ (i\varepsilon B - \omega I)u = 0 \]

\[ \omega = EV(i\varepsilon B) = i\varepsilon \cdot EV(B) \]

\[ \Rightarrow B \text{ needs to have only negative (real) eigenvalues.} \]
Stability of Hyperbolic Relaxation System

\[ \partial_t u + A \partial_x u = \varepsilon B u \]

Wave ansatz

\[ u = u_0 \cdot e^{i(kx - \omega t)}, \quad k \in \mathbb{R}, \quad \omega \in \mathbb{C}, \quad \text{Im}(\omega) \leq 0 \text{ for stability} \]

Stability analysis

\[ -i\omega u + ikAu = \varepsilon Bu \]

\[ (kA + i\varepsilon B - \omega I)u = 0 \]

\[ \omega = EV(kA + i\varepsilon B) \]

\[ \Rightarrow kA + i\varepsilon B \text{ needs to have only eigenvalues with negative imaginary part} \]
Instability of HME, [Zhao, Luo et al., 2015]

’’Stability Analysis of a Globally Hyperbolic Moment System in One Dimension’’

HME with BGK

\[ \partial_t w + D^{-1} M D \partial_x w = \varepsilon B w \]

Example: \( n = 4 \)

Existence of eigenvalue with positive imaginary part and breakdown of numerical simulation.
Towards Stable Hyperbolic Moment Equations

HME with BGK

\[ \partial_t w + D^{-1} MD \partial_x w = \varepsilon B w \]

Stability analysis

\[ \omega = EV(kD^{-1} MD + i\varepsilon B) \]
\[ = EV(kM + i\varepsilon DBD^{-1}) \]
\[ = EV(kM + i\varepsilon BD^{-1}) \]
Towards Stable Hyperbolic Moment Equations

HME with BGK

\[ \partial_t w + D^{-1}MD \partial_x w = \varepsilon Bw \]

Stability analysis

\[ \omega = EV(kD^{-1}MD + i\varepsilon B) \]
\[ = EV(kM + i\varepsilon DBD^{-1}) \]
\[ = EV(kM + i\varepsilon BD^{-1}) \]

Stable for \( D = \text{diag}(d_{ii})! \)
### SHME explanation

#### Boltzmann equation

\[
\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = 0
\]

\[
f = f_\alpha H_\alpha \Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f_\alpha}{\partial s} H_\alpha + f_\alpha \frac{\partial H_\alpha}{\partial s}, \ s = t, x
\]

#### Derivative relation for weighted Hermite polynomials

\[
\frac{\partial H_\alpha}{\partial s} = \frac{\partial u}{\partial s} H_{\alpha + 1} + \frac{1}{2} \frac{\partial \theta}{\partial s} H_{\alpha + 2}
\]

#### Recurrence relation for weighted Hermite polynomials

\[
\xi H_\alpha = \theta H_{\alpha + 1} + u H_\alpha + \alpha H_{\alpha - 1}
\]
**Grad's Equations**

\[ \partial_t \mathbf{w} + \mathbf{A}_{\text{Grad}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{Bw} \]

**Grad model**

\[
\mathbf{A}_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]
Hyperbolic Moment Equations

\[ \partial_t \mathbf{w} + \mathbf{A}_{\text{HME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w} \]

\[
\mathbf{A}_{\text{HME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & \nu
\end{pmatrix}
\]
Quadrature-Based Moment Equations

\[ \partial_t w + A_{QBME} \partial_x w = \frac{1}{\tau} Bw \]

**QBME model**

\[
A_{QBME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10f_4}{\theta} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho\theta} & \nu \\
\end{pmatrix}
\]
Stable Hyperbolic Moment Equations

\[
\partial_t w + A_{SHME} \partial_x w = \frac{1}{\tau} B w
\]

SHME model

\[
A_{SHME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\theta & \nu & 1 & 0 & 0 \\
\rho & \nu & 2\theta & \nu & 0 \\
0 & \rho\theta & 0 & \nu & 4 \\
0 & 0 & \frac{\rho\theta}{2} & \theta & \nu
\end{pmatrix}
\]
Simulation Results
Shock Tube Test Case

\[
\frac{\partial}{\partial t} \mathbf{w} + A(\mathbf{w}) \frac{\partial}{\partial x} \mathbf{w} = -1 \tau \mathbf{Pw},
\]

\(x \in [-2, 2]\)

\(\rho_L, u_L, \theta_L\)

\(\rho_R, u_R, \theta_R\)
Riemann problem with BGK collision operator

\[ \partial_t \mathbf{w} + \mathbf{A}(\mathbf{w}) \partial_x \mathbf{w} = -\frac{1}{\tau} \mathbf{P} \mathbf{w}, \quad x \in [-2, 2] \]

\[ \rho_L = 7, \rho_R = 1 \]

- Variable vector \( \mathbf{w} = (\rho, u, \theta, f_3, f_4) \)
- Relaxation time \( \tau = \frac{Kn}{\rho} \Rightarrow \text{non-linear} \)
Grad vs QBME, $Kn = 0.05$

![Graph showing comparison between Grad and QBME models for $Kn = 0.05$. The graph plots density ($\rho$), pressure ($p$), and velocity ($u$) against a parameter, with clear distinctions between Grad and QBME models.](image)
HME vs QBME, $Kn = 0.5$
HME vs SHME, $Kn = 0.5$

![Graph showing comparison between HME and SHME for different variables including density, pressure, and velocity. The graph illustrates the stability of Hyperbolic Moment Equations (HME) and SHME models.]
HME vs SHME, $f_3$, $f_4$
Summary

(1) Stability of hyperbolic moment equations
(2) Instability of existing models
(3) New stable model: SHME
(4) Promising numerical results

Further Work
- Derivation of further stable moment models
- More simulations and test cases
Summary

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Further Work

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Thank you for your attention!