On the Derivation and Numerical Solution of Stable Hyperbolic Moment Equations

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16th International Conference on Hyperbolic Problems, Aachen
August 1st, 2016

Hyperbolic Moment Models

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t,\mathbf{x},\mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t,\mathbf{x},\mathbf{c}) = S(f)$$

PDE for particles' *probability density function* $f(t,\mathbf{x},\mathbf{c})$

- Describes change of f due to transport and collisions
- Collision operator S
- Usually a 7-dimensional phase space

Moment Models for Boltzmann Equation

$$\frac{\partial}{\partial t} f(t,x,c) + c\frac{\partial}{\partial x} f(t,x,c) = S(f)$$

Replacement to reduce complexity

$$f(t,x,c) \longleftrightarrow \mathbf{w}(t,x) = (\rho, u, \theta, f_i)^T$$

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$$\Downarrow$$

Moment equations

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\frac{\partial}{\partial x}\mathbf{w} = \frac{1}{\tau}\mathbf{B}\mathbf{w}$$

Grad's Method [Grad, 1949]

Standard Galerkin projection of equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{Grad}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

Grad model ($M = 4$)

$$\mathbf{A}_{\mathrm{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix}$$

not globally hyperbolic

Hyperbolic Moment Equations [Cai et al., 2012]

Modify last equation to achieve hyperbolicity

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{HME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

HME model ($M = 4$)

$$\mathbf{A}_{\mathrm{HME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & \color{red}{0} & \color{red}{-f_3} & \theta & v \end{pmatrix}$$

globally hyperbolic

Quadrature-Based Moment Equations [JK, 2013]

Use of Gaussian quadrature

$$\partial_t \mathbf{w} + \mathbf{A}_{\text{QBME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

QBME model ($M = 4$)

$$\mathbf{A}_{\text{QBME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10 f_4}{\theta} & v & 4 \\ -\frac{f_3 \theta}{\rho} & 5 f_4 & -f_3 & \theta + \frac{15 f_4}{\rho\theta} & v \end{pmatrix}$$

globally hyperbolic

Stability Analysis

Stability Analysis

Necessary condition

- compute dispersion relation from wave ansatz
- condition to prevent blow-up of solution

Sufficient condition [Yong, 1999]

- negative semi-definite source term
- symmetrizable hyperbolic term
- coupling condition of hyperbolic and source term

Necessary Condition

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \varepsilon \mathbf{B}\mathbf{u}$$

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Wave ansatz

$$\mathbf{u} = \mathbf{u}_0 \cdot e^{i(kx-\omega t)}, k \in \mathbb{R}, \omega \in \mathbb{C}, \quad Im(\omega) \leq 0 \text{ for stability}$$

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Stability analysis

$$\begin{aligned} -i\omega\mathbf{u} + ik\mathbf{A}\mathbf{u} &= \varepsilon\mathbf{B}\mathbf{u} \\ \Rightarrow (k\mathbf{A} + i\varepsilon\mathbf{B} - \omega\mathbf{I})\,\mathbf{u} &= \mathbf{0} \end{aligned}$$

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$$\omega = EV(k\mathbf{A} + i\varepsilon\mathbf{B})$$

$\Rightarrow$ all eigenvalues of $k\mathbf{A} + i\varepsilon\mathbf{B}$ need to have negative imaginary part

Instability of HME [Zhao, Yong et al.]

"Stability Analysis of a Globally Hyperbolic Moment System in One Dimension"

Hyperbolic Moment System

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} = \varepsilon \mathbf{B}\mathbf{w}$$

Example:

Existence of eigenvalue with positive imaginary part and breakdown of numerical simulation.

Towards Stable Hyperbolic Moment Equations

HME system

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} = \varepsilon \mathbf{B}\mathbf{w}$$

Linear stability analysis

$$\begin{aligned} \omega &= EV(k\mathbf{A} + i\varepsilon\mathbf{B}) \\ \Rightarrow \omega &= EV(k\mathbf{D}^{-1}\mathbf{M}\mathbf{D} + i\varepsilon\mathbf{B}) \end{aligned}$$

Towards Stable Hyperbolic Moment Equations

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Stable for $\mathbf{D} = diag(d_{ii})$!

Stable Hyperbolic Moment Equations

Modify matrix to achieve stability

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{SHME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

SHME model (M=4)

$$\mathbf{A}_{\mathrm{SHME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 0 & \frac{\rho\theta}{2} & v & 4 \\ 0 & 0 & 0 & \theta & v \end{pmatrix}$$

globally hyperbolic and stable

Sufficient Condition [Yong, 1999]

HME system

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} = \varepsilon \mathbf{B}\mathbf{w}$$

1. negative semi-definite source term

- $\mathbf{B} \leq \mathbf{0}$
-

2. symmetrizable hyperbolic term

- $\exists$ symm. $\mathbf{R}: \quad \mathbf{R} \cdot \mathbf{A} = \mathbf{A}^T \cdot \mathbf{R}$
-

3. coupling condition of hyperbolic and source term

- $\mathbf{R} \cdot \mathbf{B} + \mathbf{B}^T \cdot \mathbf{R} \leq \mathbf{0}$
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- $\mathbf{B} \leq \mathbf{0}$
- $\mathbf{B} = diag(0, 0, 0, -1, \dots, -1) \quad \Rightarrow \checkmark$

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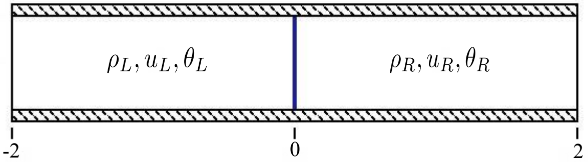
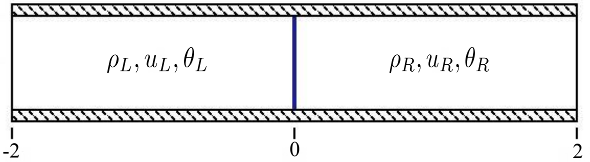
- $\mathbf{R} = \mathbf{D}^T\mathbf{D} \Rightarrow \mathbf{D}^T \cdot \mathbf{B} + \mathbf{B}^T \cdot \mathbf{D} \leq \mathbf{0} \quad \Rightarrow \checkmark$ for diagonal $\mathbf{D}$

Simulation Results

Shock Tube Test Case



Shock Tube Test Case



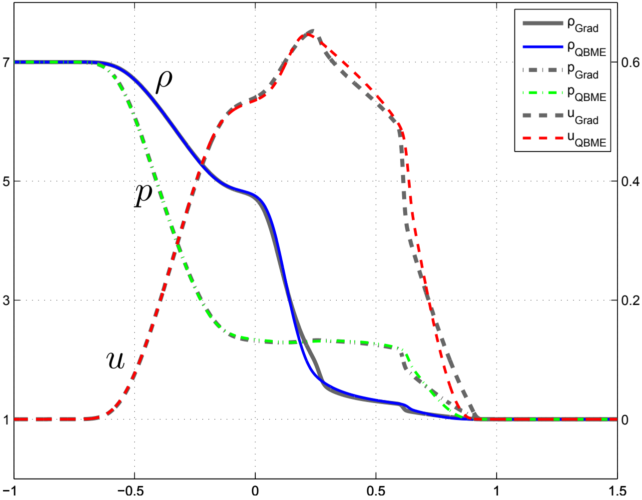
Riemann problem

$$\partial_t \mathbf{w} + \mathbf{A}(\mathbf{w})\partial_x \mathbf{w} = \frac{1}{\tau}\mathbf{B}\mathbf{w}, \quad x \in [-2, 2]$$

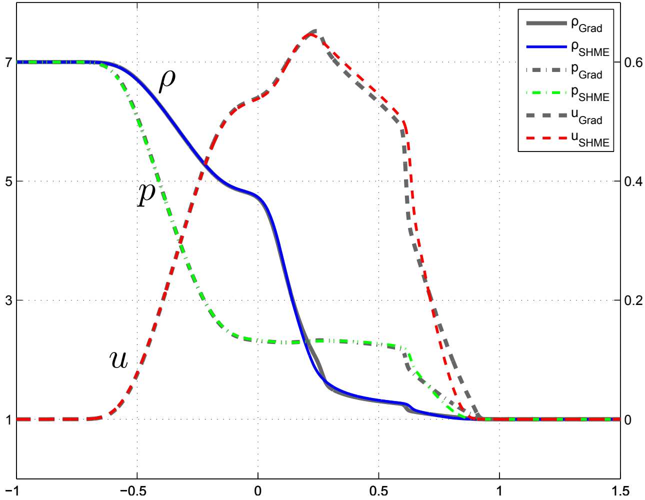
$$\rho_L = 7, \rho_R = 1$$

- Variable vector $\mathbf{w} = (\rho, u, \theta, f_3, f_4)$
- Relaxation time $\tau = \frac{\mathrm{Kn}}{\rho} \Rightarrow$ non-linear

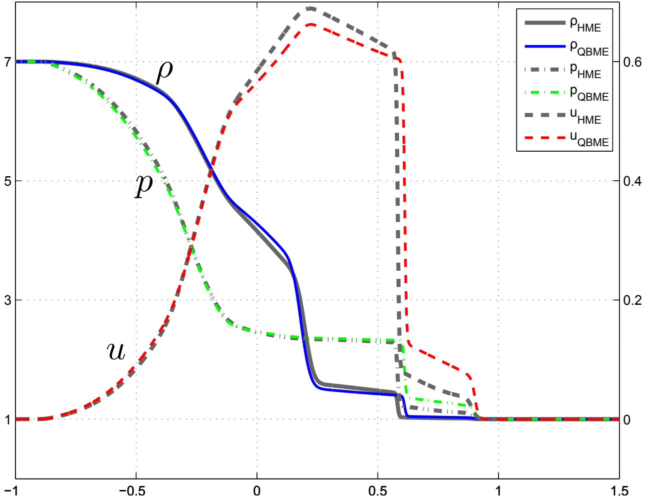
Grad vs QBME, $\mathrm{Kn} = 0.05$



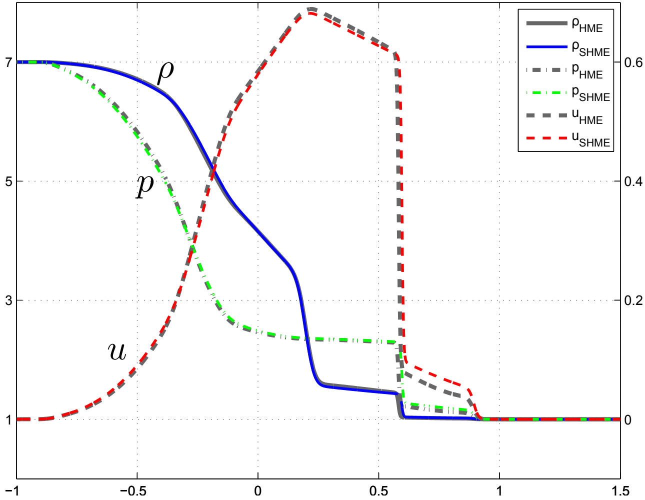
Grad vs SHME, $\mathrm{Kn} = 0.05$



HME vs QBME, $\mathrm{Kn} = 0.5$



HME vs SHME, $\mathrm{Kn} = 0.5$



Conclusion

Summary

- Hyperbolic moment models
- Stability analysis
- Numerical results

Further Work

- Derivation of other stable models
- Numerical simulations

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Thank you for your attention!

References

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