Numerical Solution of Hyperbolic Moment Models for the Boltzmann Equation

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Introduction

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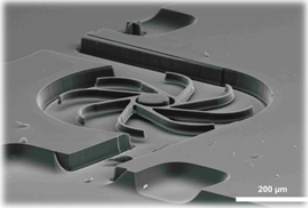
Aim

Derive hyperbolic PDE systems for rarefied gas flows

Extension of standard fluid dynamic equations

- Reentry flows
- Micro channel flows



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Importance of Hyperbolicity

- Physical solutions with bounded propagation speeds
- Well-posedness and stability of the solution

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

PDE for particles' *probability density function* $f(t, \mathbf{x}, \mathbf{c})$

- Describes change of f due to transport and collisions
- Collision operator S
- Usually a 7-dimensional phase space

Model Order Reduction

Ansatz

$$f(t,\mathbf{x},\mathbf{c}) = \sum_{i=0}^{M} f_i(t,\mathbf{x})\mathcal{H}_i^{\rho,\mathbf{v},\theta}(\mathbf{c})$$

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Reduction of Complexity

One PDE for $f(t,\mathbf{x},\mathbf{c})$ that is 7-dimensional

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Reduction of Complexity

One PDE for $f(t,\mathbf{x},\mathbf{c})$ that is 7-dimensional

$$\Downarrow$$

System of PDEs for $\rho(t,\mathbf{x}), \mathbf{v}(t,\mathbf{x}), \theta(t,\mathbf{x}), f_i(t,\mathbf{x})$ that is 4-dimensional

Hyperbolic Moment Equations

Operator Projection Framework (OP) [JK et al., 2015]

Boltzmann equation

$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0$$

$$\Downarrow$$

Operator Projection Framework

$$\mathbf{D} \frac{\partial}{\partial t} \mathbf{w} + \mathbf{M}\mathbf{D} \frac{\partial}{\partial x} \mathbf{w} = \mathbf{0}$$

Operator Projection Framework (OP) [JK et al., 2015]

Boltzmann equation

$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0$$

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Operator Projection Framework

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\frac{\partial}{\partial x}\mathbf{w} = \mathbf{0}$$

GRAD's Equations [GRAD 1949]

Standard Galerkin projection of equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{Grad}} \partial_x \mathbf{w} = 0$$

Grad model

$$\mathbf{A}_{\mathrm{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix}$$

Grad's Equations [Grad 1949]

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$\Rightarrow$ not globally hyperbolic

Hyperbolic Moment Equations [Cai et al. 2012]

Modify last equation to achieve hyperbolicity

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{HME}} \partial_x \mathbf{w} = 0$$

HME model

$$\mathbf{A}_{\mathrm{HME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & \color{red}{0} & \color{red}{-f_3} & \theta & v \end{pmatrix}$$

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$\Rightarrow$ globally hyperbolic

Quadrature-Based Moment Equations [JK 2013]

Substitute integration by Gaussian quadrature to achieve hyperbolicity

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{QBME}} \partial_x \mathbf{w} = 0$$

QBME model

$$\mathbf{A}_{\mathrm{QBME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10 f_4}{\theta} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15 f_4}{\rho\theta} & v \end{pmatrix}$$

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$\Rightarrow$ globally hyperbolic

Model Summary

Properties

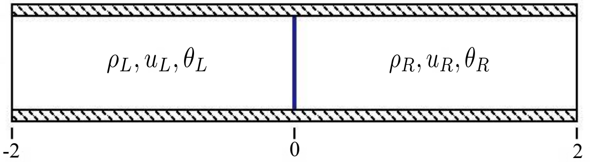
- globally hyperbolic system
- multiple spatial dimensions
- rotational invariance
- single framework includes all theories

Questions

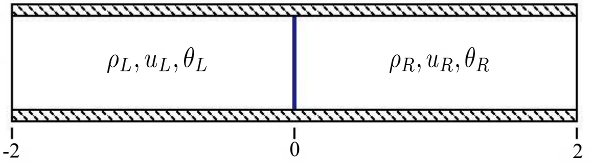
- accuracy of new models
- numerical simulations

Numerical Simulations

Shock Tube Test Case



Shock Tube Test Case



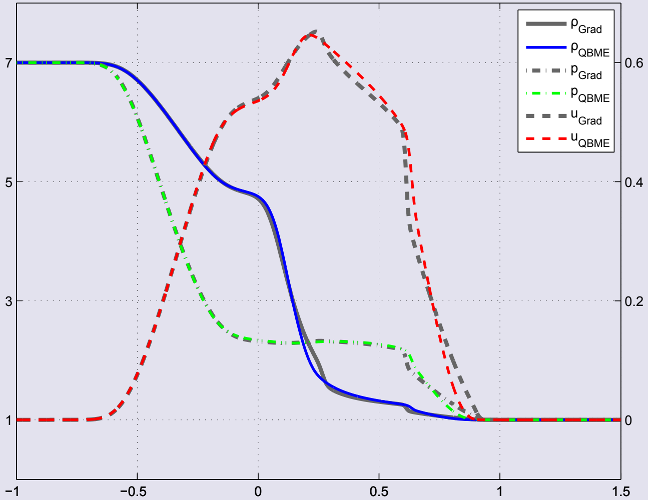
Riemann problem with BGK collision operator

$$\partial_t \mathbf{w} + \mathbf{A}\partial_x \mathbf{w} = -\frac{1}{\tau}\mathbf{P}\mathbf{w}, \quad x \in [-2, 2]$$

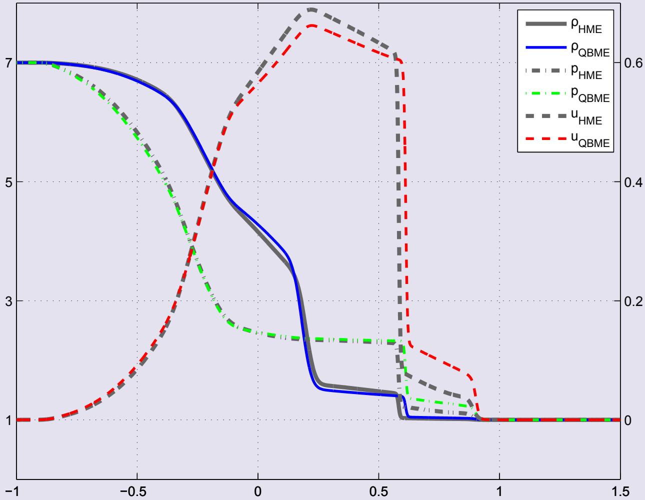
$$\rho_L = 7, \rho_R = 1$$

- Variable vector $\mathbf{w} = (\rho, v, \theta, f_3, f_4)$
- Relaxation time $\tau = \frac{\mathrm{Kn}}{\rho} \Rightarrow$ non-linear

QBME vs Grad, $\mathrm{Kn} = 0.05$



QBME vs HME, $\mathrm{Kn} = 0.5$



Summary and Further Work

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- OP, HME, QBME
- Numerical solutions

Further Work

- Tests with more equations
- Different test cases
- 2D simulations

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Thank you for your attention!

References

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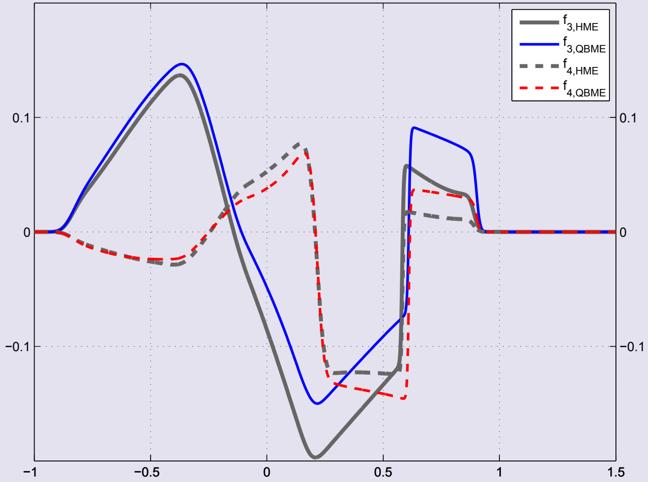
QBME vs HME, f_3, f_4



PRICE-C scheme [Canestrelli, 2009]

First order scheme

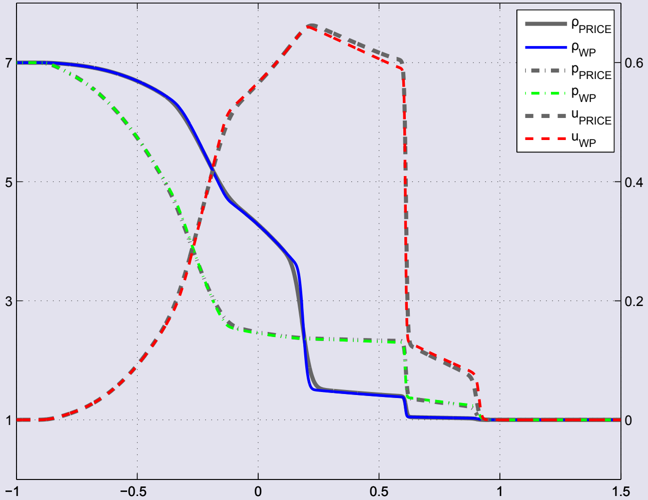
$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^-_{i+\frac{1}{2}}\left(\mathbf{u}_{i+1}^n - \mathbf{u}_i^n\right) + \mathbf{A}^+_{i-\frac{1}{2}}\left(\mathbf{u}_i^n - \mathbf{u}_{i-1}^n\right)\right)$$

- PRImitive CEntered scheme, uses no eigenvalue information
- Reduces to FORCE scheme in the conservative case

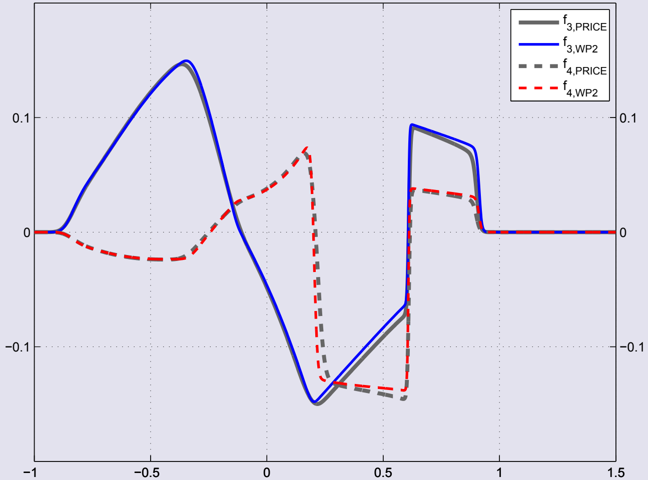
FORCE scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^{\mathrm{FORCE}}_{\mathbf{i+\frac{1}{2}}} - \mathbf{F}^{\mathrm{FORCE}}_{\mathbf{i-\frac{1}{2}}}\right)$$

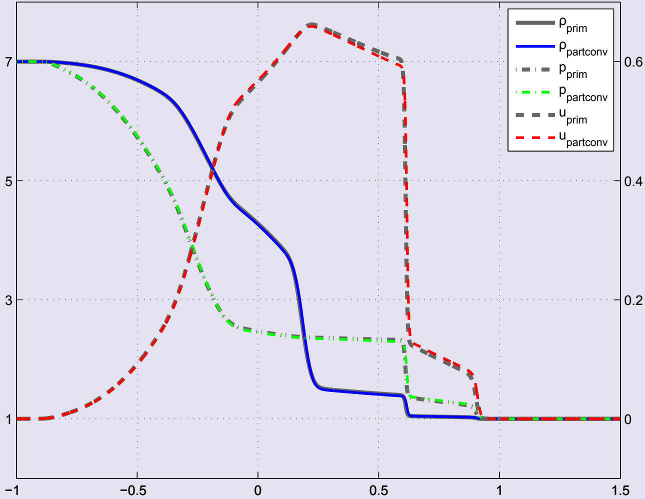
PRICE vs WP, $\mathrm{Kn} = 0.5$



PRICE vs WP, f_3, f_4



Primitive vs Partially Conserved, $\mathrm{Kn} = 0.5$



Primitive vs Partially Conserved, f_3, f_4