Numerical Solution of Hyperbolic Moment Models for the Boltzmann Equation

Julian Koellermeier, Manuel Torrilhon

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Introduction
Introduction

**Aim**

Derive hyperbolic PDE systems for rarefied gas flows

**Extension of standard fluid dynamic equations**

- Reentry flows
- Micro channel flows
Introduction

Aim

Derive hyperbolic PDE systems for rarefied gas flows

Extension of standard fluid dynamic equations

- Reentry flows
- Micro channel flows

Importance of Hyperbolicity

- Physical solutions with bounded propagation speeds
- Well-posedness and stability of the solution
Boltzmann Transport Equation

\[
\frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f)
\]

PDE for particles’ *probability density function* \( f(t, x, c) \)

- Describes change of \( f \) due to transport and collisions
- Collision operator \( S \)
- Usually a 7-dimensional phase space
Model Order Reduction

Ansatz

\[ f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) \mathcal{H}_i^{\rho, v, \theta}(c) \]
Model Order Reduction

Ansatz

\[ f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) H_i^{\rho, v, \theta}(c) \]

Reduction of Complexity

One PDE for \( f(t, x, c) \) that is 7-dimensional
**Model Order Reduction**

**Ansatz**

\[ f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) H_i^{\rho, \nu, \theta}(c) \]

**Reduction of Complexity**

One PDE for \( f(t, x, c) \) that is 7-dimensional

\[ \Downarrow \]

System of PDEs for \( \rho(t, x), \nu(t, x), \theta(t, x), f_i(t, x) \) that is 4-dimensional
Hyperbolic Moment Equations
Boltzmann equation

\[ \frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0 \]

Operator Projection Framework

\[ D \frac{\partial}{\partial t} w + MD \frac{\partial}{\partial x} w = 0 \]
Boltzmann equation

\[
\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0
\]

Operator Projection Framework

\[
\frac{\partial}{\partial t} w + D^{-1} MD \frac{\partial}{\partial x} w = 0
\]
\[ \frac{\partial t w + A_{\text{Grad}} \partial_x w = 0}{} \]

Grad model

\[ A_{\text{Grad}} = \begin{pmatrix}
 v & \rho & 0 & 0 & 0 \\
 \frac{\theta}{\rho} & v & 1 & 0 & 0 \\
 0 & 2\theta & v & \frac{6}{\rho} & 0 \\
 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\
 -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v
\end{pmatrix} \]
**Grad's Equations [Grad 1949]**

### Standard Galerkin projection of equations

\[ \partial_t w + A_{\text{Grad}} \partial_x w = 0 \]

### Grad model

\[
A_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho \theta}{2} & \nu & 4 \\
-\frac{f_3 \theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]

\[ \Rightarrow \text{not globally hyperbolic} \]
Modify last equation to achieve hyperbolicity

\[ \partial_t w + A_{\text{HME}} \partial_x w = 0 \]

HME model

\[
A_{\text{HME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & \nu
\end{pmatrix}
\]
Modify last equation to achieve hyperbolicity

\[
\partial_t w + A_{HME} \partial_x w = 0
\]

HME model

\[
A_{HME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{\rho} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & \nu \\
\end{pmatrix}
\]

⇒ globally hyperbolic
Substitute integration by Gaussian quadrature to achieve hyperbolicity

\[
\partial_t w + A_{QBME} \partial_x w = 0
\]

QBME model

\[
A_{QBME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10f_4}{\theta} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho\theta} & \nu
\end{pmatrix}
\]
Substitute integration by Gaussian quadrature to achieve hyperbolicity

\[ \partial_t w + A_{QBME} \partial_x w = 0 \]

**QBME model**

\[
A_{QBME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\theta & v & 1 & 0 & 0 \\
0 & 2\theta & v & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho \theta}{2} & -\frac{10f_4}{\theta} & v \\
-\frac{f_3 \theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho \theta} & v \\
\end{pmatrix}
\]

\[ \Rightarrow \text{globally hyperbolic} \]
Model Summary

Properties
- globally hyperbolic system
- multiple spatial dimensions
- rotational invariance
- single framework includes all theories

Questions
- accuracy of new models
- numerical simulations
Numerical Simulations
Shock Tube Test Case

\begin{align*}
\frac{\partial}{\partial t} w + \frac{\partial}{\partial x} A w &= -\frac{1}{\tau} Pw, \\
\rho_{L}, u_{L}, \theta_{L} &
\end{align*}

\rho_{L} = 7, \rho_{R} = 1

Variable vector \( w = (\rho, v, \theta, f_{3}, f_{4}) \)

Relaxation time \( \tau = \frac{\rho}{K_{n}} \Rightarrow \text{non-linear} \)

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Riemann problem with BGK collision operator

\[ \partial_t w + A \partial_x w = -\frac{1}{\tau} Pw, \quad x \in [-2, 2] \]

\[ \rho_L = 7, \rho_R = 1 \]

- Variable vector \( w = (\rho, \nu, \theta, f_3, f_4) \)
- Relaxation time \( \tau = \frac{Kn}{\rho} \Rightarrow \text{non-linear} \)
QBME vs Grad, $Kn = 0.05$
QBME vs HME, $Kn = 0.5$
Summary

- OP, HME, QBME
- Numerical solutions

Further Work

- Tests with more equations
- Different test cases
- 2D simulations
Summary and Further Work

Summary
- OP, HME, QBME
- Numerical solutions

Further Work
- Tests with more equations
- Different test cases
- 2D simulations

Thank you for your attention!
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QBME vs HME, $f_3, f_4$
**PRICE-C scheme [Canestrelli, 2009]**

**First order scheme**

\[
\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x} \left( A_i^{\frac{1}{2}+} (\mathbf{u}_{i+1}^n - \mathbf{u}_i^n) + A_i^{\frac{1}{2}-} (\mathbf{u}_i^n - \mathbf{u}_{i-1}^n) \right)
\]

- PRImitive CEntered scheme, uses no eigenvalue information
- Reduces to FORCE scheme in the conservative case

**FORCE scheme**

\[
\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x} \left( \mathbf{F}_{i+\frac{1}{2}}^{\text{FORCE}} - \mathbf{F}_{i-\frac{1}{2}}^{\text{FORCE}} \right)
\]
PRICE vs WP, Kn = 0.5
PRICE vs WP, $f_3$, $f_4$
Primitive vs Partially Conserved, $Kn = 0.5$
Primitive vs Partially Conserved, $f_3$, $f_4$