Numerical Convergence Study of Hyperbolic Moment Models in Partially-Conservative Form

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Outline

1. Introduction to Moment Methods
2. Numerical Methods
3. Simulation Results
Boltzmann Transport Equation

\[
\frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f)
\]

PDE for particles’ probability density function \( f(t, x, c) \)

- Describes change of \( f \) due to transport and collisions
- Collision operator \( S \)
- Usually a 7-dimensional phase space
Model Reduction (1D)

\[ \frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = S(f) \]

Replacement to reduce complexity

\[ f(t, x, c) \leftrightarrow w(t, x) \]
Model Reduction (1D)

\[ \frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = S(f) \]

Replacement to reduce complexity

\[ f(t, x, c) \longleftrightarrow \mathbf{w}(t, x) \]

Challenge

Highly non-linear ansatz is necessary for hypersonic problems
Model Reduction (1D)

\[ \frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = S(f) \]

Replacement to reduce complexity

\[ f(t, x, c) \leftrightarrow w(t, x) \]

Challenge

Highly non-linear ansatz is necessary for hypersonic problems

Model or closure relation

\[ f(t, x, c) = \sum_{i=0}^{M} f_i(t, x) H_i \left( \frac{c - v}{\sqrt{\theta}} \right), \quad w(t, x) = (\rho, v, \theta, f_i)^T \]
Moment Equations

Boltzmann equation $\Rightarrow$ moment equations

$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = S(f) \quad \Rightarrow \quad D \frac{\partial}{\partial t} w + MD \frac{\partial}{\partial x} w = -\frac{1}{\tau} B w$$
Boltzmann equation $\Rightarrow$ moment equations

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = S(f) \quad \Rightarrow \quad \frac{\partial}{\partial t} w + D^{-1}MD \frac{\partial}{\partial x} w = -\frac{1}{\tau} D^{-1} B w$$
Moment Equations

Boltzmann equation $\Rightarrow$ moment equations

\[
\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = S(f) \quad \Rightarrow \quad \frac{\partial}{\partial t} w + D^{-1} M D \frac{\partial}{\partial x} w = -\frac{1}{\tau} D^{-1} B w
\]

Operator Projection Framework \([\text{FAN, JK et al.}, 2016]\)

- $\tilde{D} = PDP^T$, $\tilde{M} = PMP^T$
- Projection matrix $P$ depends on the model
- Projections guarantee hyperbolicity
**Grad’s Method** \([\text{Grad}, 1949]\)

**Standard Galerkin projection of equations**

\[
\partial_t \mathbf{w} + \mathbf{A}_{\text{Grad}} \partial_x \mathbf{w} = -\frac{1}{\tau} \mathbf{Bw}
\]

**Grad model \((M = 4)\)**

\[
\mathbf{A}_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\theta & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]

Not globally hyperbolic but in conservative form
Modify last equation to achieve hyperbolicity

$$\partial_t w + A_{HME} \partial_x w = -\frac{1}{\tau} B w$$

HME model ($M = 4$)

$$A_{HME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & \nu
\end{pmatrix}$$

Hyperbolic but only partially conservative
Use of Gaussian quadrature

\[ \partial_t w + A_{QBME} \partial_x w = -\frac{1}{\tau} B w \]

QBME model \((M = 4)\)

\[
A_{QBME} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{\theta} - \frac{10f_4}{\theta} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho\theta} & \nu
\end{pmatrix}
\]

Hyperbolic but only partially conservative
1 Introduction to Moment Methods

2 Numerical Methods

3 Simulation Results
Non-conservative Numerics

Standard conservative PDE system

\[ \partial_t u + \partial_x F(u) = 0 \]
Non-conservative Numerics

**Standard conservative PDE system**

\[ \partial_t u + \partial_x F(u) = 0 \]

**Non-conservative PDE system**

\[ \partial_t u + A(u)\partial_x u = 0 \]

- Hyperbolic moment equations cannot be written in conservative form
- At least one of the last equations is non-conservative
- Mass, momentum and energy are still conserved
Castro scheme [Castro, Pares, 2004]

First order scheme

\[
\begin{align*}
 u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x} \left( A_{i+\frac{1}{2}}^- (u_{i+1}^n - u_i^n) + A_{i-\frac{1}{2}}^+ (u_i^n - u_{i-1}^n) \right) \\
 \end{align*}
\]

- Upwind type scheme, uses eigenvalue information

\[
 A_{i+\frac{1}{2}}^\pm = A \left( u_i^n, u_{i+1}^n \right)^\pm = R \cdot \Lambda^\pm \cdot R^{-1}
\]
PRICE-C scheme [Canestrelli, 2009]

First order scheme

\[
\begin{align*}
    u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x} \left( A_{i+\frac{1}{2}}^-(u_{i+1}^n - u_i^n) + A_{i-\frac{1}{2}}^+(u_i^n - u_{i-1}^n) \right)
\end{align*}
\]

- Central scheme, adds numerical diffusion

\[
\begin{align*}
    A_{i+\frac{1}{2}}^- &= \frac{1}{4} \left( 2A(u_i^n, u_{i+1}^n) - \frac{\Delta x}{\Delta t} I - \frac{\Delta t}{\Delta x} (A(u_i^n, u_{i+1}^n))^2 \right) \\
    A_{i-\frac{1}{2}}^+ &= \frac{1}{4} \left( 2A(u_{i-1}^n, u_i^n) - \frac{\Delta x}{\Delta t} I - \frac{\Delta t}{\Delta x} (A(u_{i-1}^n, u_i^n))^2 \right)
\end{align*}
\]
Wave propagation scheme [LeVeque, 1997]

Second order scheme

\[ u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( A^+ \Delta u_i + A^- \Delta u_{i+1} \right) - \frac{\Delta t}{\Delta x} \left( \tilde{F}_{i+1} - \tilde{F}_i \right) \]

- Upwind type scheme
- Correction term for almost second order
- Only for uniform grids
1 Introduction to Moment Methods

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Shock Tube Test Case

\[ \rho_L, u_L, \theta_L \quad \text{and} \quad \rho_R, u_R, \theta_R \]

\[ \frac{\partial}{\partial t} \mathbf{w} + A(\mathbf{w}) \frac{\partial}{\partial x} \mathbf{w} = -\frac{1}{\tau} B \mathbf{w}, \quad x \in [-2, 2] \]

\[ \rho_L = 7, \rho_R = 1 \]

Variable vector \( \mathbf{w} = (\rho, u, \theta, f_3, f_4) \)

Relaxation time \( \tau = Kn \rho \Rightarrow \text{non-linear} \)
Riemann problem with BGK collision operator

\[
\partial_t \mathbf{w} + A(\mathbf{w}) \partial_x \mathbf{w} = -\frac{1}{\tau} B \mathbf{w}, \quad x \in [-2, 2]
\]

\[
\rho_L = 7, \quad \rho_R = 1
\]

- Variable vector \( \mathbf{w} = (\rho, u, \theta, f_3, f_4) \)
- Relaxation time \( \tau = \frac{Kn}{\rho} \Rightarrow \) non-linear
Method Comparison, $Kn = 0.5$
Model Comparison, $Kn = 0.05$
Model Comparison, $Kn = 0.5$
Averaged Solution $M = 8$ & $M = 9$, $Kn = 0.5$

![Graph showing the comparison between QBME8+9 and DVM methods for averaged solution $M = 8$ and $M = 9$ with $Kn = 0.5$. The graph plots density $\rho$, pressure $p$, and velocity $u$ against a scaled horizontal axis. The red line represents QBME8+9, and the black line represents DVM. The x-axis is scaled from -1 to 1.5, and the y-axis shows density and pressure values.]

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Convergence of Model, $Kn = 0.5$

![Graph showing the convergence of different variables for QBME and HME models for $Kn = 0.5$. The graph includes lines for variables $\rho$, $u$, $p$, and $\theta$, as well as a line for the 1st order model.]
Conclusion

Summary
- Hyperbolic moment equations
- Non-conservative numerics
- Simulation results

Further work
- Derivation of new models
- Higher order schemes
- 2D simulations
Summary

- Hyperbolic moment equations
- Non-conservative numerics
- Simulation results

Further work

- Derivation of new models
- Higher order schemes
- 2D simulations

Thank you for your attention!
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