Numerical Convergence Study of Hyperbolic Moment Models in Partially-Conservative Form

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Outline

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

PDE for particles' *probability density function* $f(t, \mathbf{x}, \mathbf{c})$

- Describes change of f due to transport and collisions
- Collision operator S
- Usually a 7-dimensional phase space

Model Reduction (1D)

$$\frac{\partial}{\partial t} f(t,x,c) + c\frac{\partial}{\partial x} f(t,x,c) = S(f)$$

Replacement to reduce complexity

$$f(t,x,c) \longleftrightarrow \mathbf{w}(t,x)$$

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Challenge

Highly non-linear ansatz is necessary for hypersonic problems

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Model or closure relation

$$f(t,x,c) = \sum_{i=0}^{M} f_i(t,x) H_i\left(\frac{c-v}{\sqrt{\theta}}\right), \quad \mathbf{w}(t,x) = (\rho, v, \theta, f_i)^T$$

Moment Equations

Boltzmann equation $\Longrightarrow$ moment equations

$$\frac{\partial}{\partial t}f + c\frac{\partial}{\partial x}f = S(f) \quad \Longrightarrow \quad \mathbf{D}\frac{\partial}{\partial t}\mathbf{w} + \mathbf{M}\mathbf{D}\frac{\partial}{\partial x}\mathbf{w} = -\frac{1}{\tau}\mathbf{B}\mathbf{w}$$

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$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = S(f) \quad \Longrightarrow \frac{\partial}{\partial t}\mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\frac{\partial}{\partial x}\mathbf{w} = -\frac{1}{\tau}\mathbf{D}^{-1}\mathbf{B}\mathbf{w}$$

Moment Equations

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Operator Projection Framework [Fan, JK et al., 2016]

- $\widetilde{\mathbf{D}} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, $\widetilde{\mathbf{M}} = \mathbf{P}\mathbf{M}\mathbf{P}^T$
- Projection matrix **P** depends on the model
- Projections guarantee hyperbolicity

Grad's Method [Grad, 1949]

Standard Galerkin projection of equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{Grad}} \partial_x \mathbf{w} = -\frac{1}{\tau}\mathbf{B}\mathbf{w}$$

Grad model ($M = 4$)

$$\mathbf{A}_{\mathrm{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix}$$

Not globally hyperbolic but in conservative form

Hyperbolic Moment Equations [Cai et al., 2012]

Modify last equation to achieve hyperbolicity

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{HME}} \partial_x \mathbf{w} = -\frac{1}{\tau}\mathbf{B}\mathbf{w}$$

HME model ($M = 4$)

$$\mathbf{A}_{\mathrm{HME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & \color{red}{0} & \color{red}{-f_3} & \theta & v \end{pmatrix}$$

Hyperbolic but only partially conservative

Quadrature-Based Moment Equations [JK, 2013]

Use of Gaussian quadrature

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{QBME}} \partial_x \mathbf{w} = -\frac{1}{\tau} \mathbf{B} \mathbf{w}$$

QBME model ($M = 4$)

$$\mathbf{A}_{\mathrm{QBME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10 f_4}{\theta} & v & 4 \\ -\frac{f_3 \theta}{\rho} & 5 f_4 & -f_3 & \theta + \frac{15 f_4}{\rho\theta} & v \end{pmatrix}$$

Hyperbolic but only partially conservative

Non-conservative Numerics

Standard conservative PDE system

$$\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{0}$$

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Non-conservative PDE system

$$\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u} = \mathbf{0}$$

- Hyperbolic moment equations cannot be written in conservative form
- At least one of the last equations is non-conservative
- Mass, momentum and energy are still conserved

Castro scheme [Castro, Pares, 2004]

First order scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^-_{i+\frac{1}{2}}\left(\mathbf{u}_{i+1}^n - \mathbf{u}_i^n\right) + \mathbf{A}^+_{i-\frac{1}{2}}\left(\mathbf{u}_i^n - \mathbf{u}_{i-1}^n\right)\right)$$

- Upwind type scheme, uses eigenvalue information

$$\mathbf{A}^{\pm}_{i+\frac{1}{2}} = \mathbf{A}\left(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n\right)^{\pm} = \mathbf{R}\cdot\mathbf{\Lambda}^{\pm}\cdot\mathbf{R}^{-1}$$

PRICE-C scheme [Canestrelli, 2009]

First order scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^-_{i+\frac{1}{2}}\left(\mathbf{u}_{i+1}^n - \mathbf{u}_i^n\right) + \mathbf{A}^+_{i-\frac{1}{2}}\left(\mathbf{u}_i^n - \mathbf{u}_{i-1}^n\right)\right)$$

- Central scheme, adds numerical diffusion

$$\begin{aligned}
\mathbf{A}^-_{i+\frac{1}{2}} &= \frac{1}{4}\left(2\mathbf{A}\left(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n\right) - \frac{\Delta x}{\Delta t}\mathbf{I} - \frac{\Delta t}{\Delta x}\left(\mathbf{A}\left(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n\right)\right)^2\right) \\
\mathbf{A}^+_{i-\frac{1}{2}} &= \frac{1}{4}\left(2\mathbf{A}\left(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n\right) - \frac{\Delta x}{\Delta t}\mathbf{I} - \frac{\Delta t}{\Delta x}\left(\mathbf{A}\left(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n\right)\right)^2\right)
\end{aligned}$$

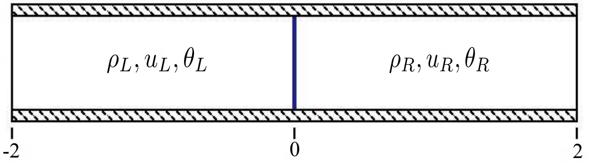
Wave propagation scheme [LeVeque, 1997]

Second order scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^+\boldsymbol{\Delta}\mathbf{u}_i + \mathbf{A}^-\boldsymbol{\Delta}\mathbf{u}_{i+1}\right) - \frac{\Delta t}{\Delta x}\left(\widetilde{\mathbf{F}}_{i+1} - \widetilde{\mathbf{F}}_i\right)$$

- Upwind type scheme
- Correction term for almost second order
- Only for uniform grids

Shock Tube Test Case



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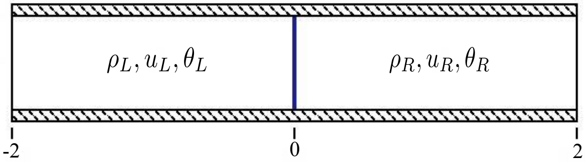


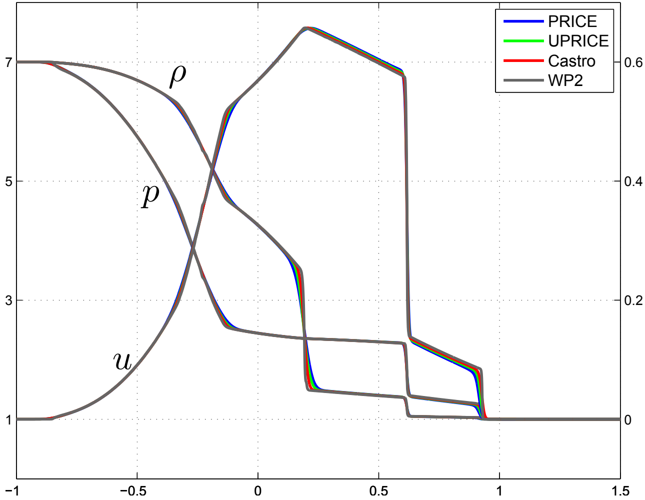
Riemann problem with BGK collision operator

$$\partial_t \mathbf{w} + \mathbf{A}(\mathbf{w})\partial_x \mathbf{w} = -\frac{1}{\tau}\mathbf{B}\mathbf{w}, \quad x \in [-2, 2]$$

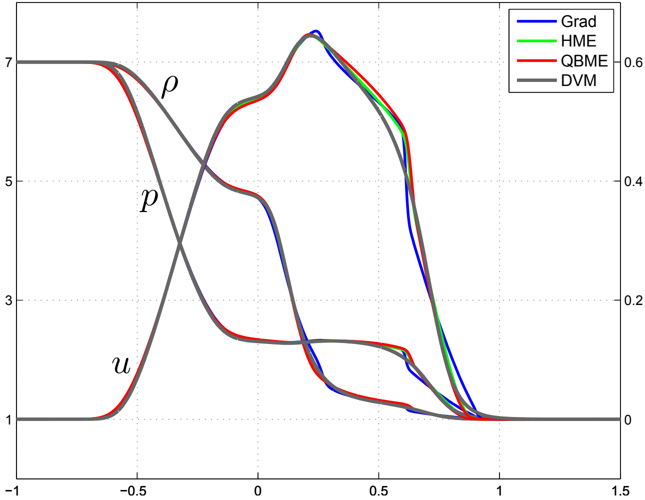
$$\rho_L = 7, \rho_R = 1$$

- Variable vector $\mathbf{w} = (\rho, u, \theta, f_3, f_4)$
- Relaxation time $\tau = \frac{\mathrm{Kn}}{\rho} \Rightarrow$ non-linear

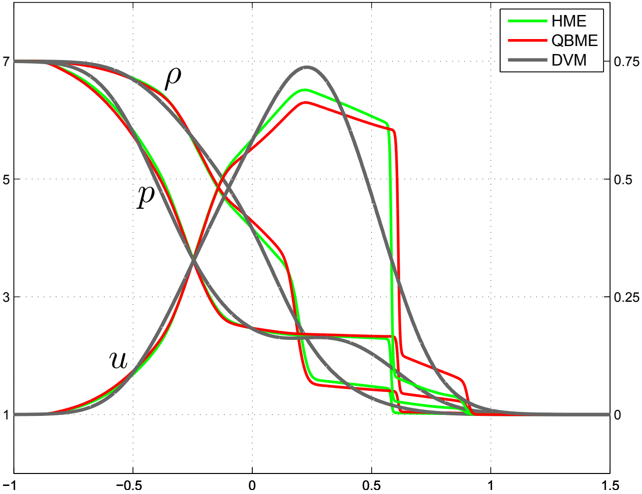
Method Comparison, $\mathrm{Kn} = 0.5$



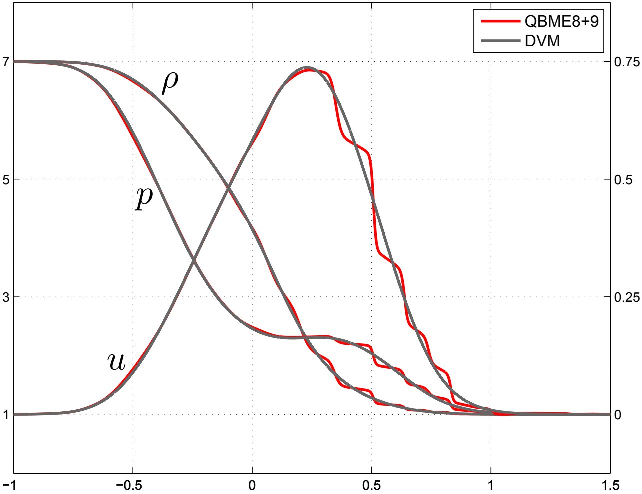
Model Comparison, $\mathrm{Kn} = 0.05$



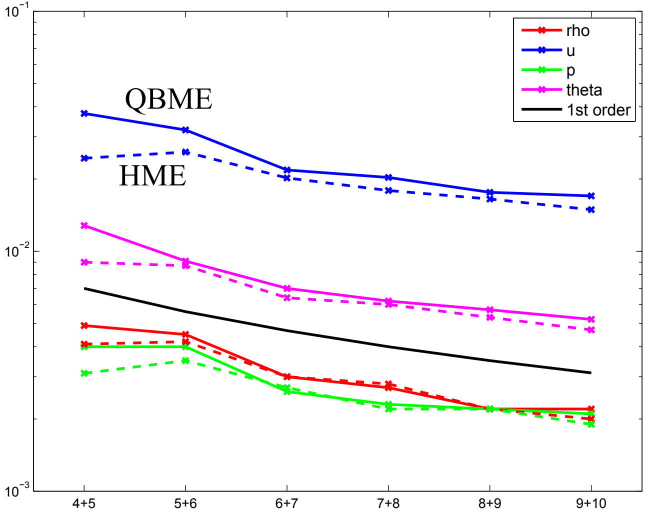
Model Comparison, $\mathrm{Kn} = 0.5$



Averaged Solution $M = 8$ & $M = 9$, $\mathrm{Kn} = 0.5$



Convergence of Model, $\mathrm{Kn} = 0.5$



Conclusion

Summary

- Hyperbolic moment equations
- Non-conservative numerics
- Simulation results

Further work

- Derivation of new models
- Higher order schemes
- 2D simulations

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Thank you for your attention!

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