Higher order numerical schemes for non-conservative hyperbolic PDEs on 2D unstructured grids

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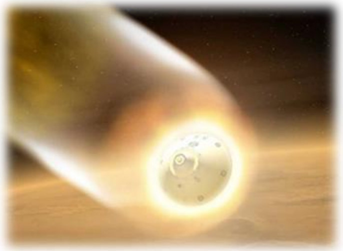
Non-conservative PDEs

- Shallow water equations
- Multi-phase flow
- **Hyperbolic models for Boltzmann equation**

  (extended fluid dynamics)

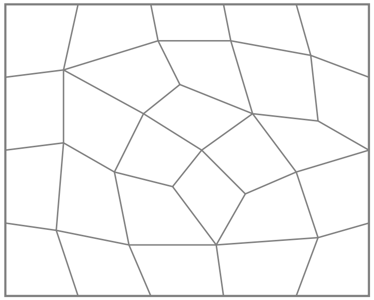


Reentry application

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \boldsymbol{A}_x(\boldsymbol{Q})\frac{\partial \boldsymbol{Q}}{\partial x} + \boldsymbol{A}_y(\boldsymbol{Q})\frac{\partial \boldsymbol{Q}}{\partial y} = \boldsymbol{0}$$

Aims:

- unstructured grids for flexibility
- 2^{nd} order scheme for accuracy



Unstructured quad-grid

Numerics

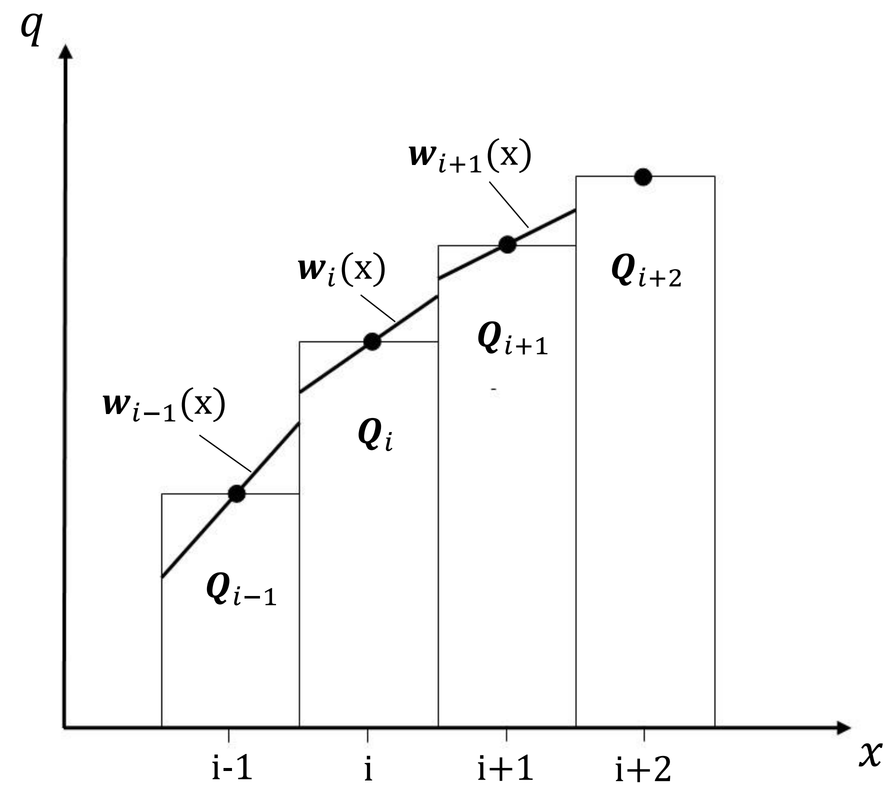
1) 1^{st} order PRICE scheme

$$\boldsymbol{Q}_i^{n+1} = \boldsymbol{Q}_i^n - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} S_j \boldsymbol{A}_{ij}^- (\boldsymbol{Q}_j^n - \boldsymbol{Q}_i^n)$$

Numerics

1) 1st order PRICE scheme

2) Reconstruction in space



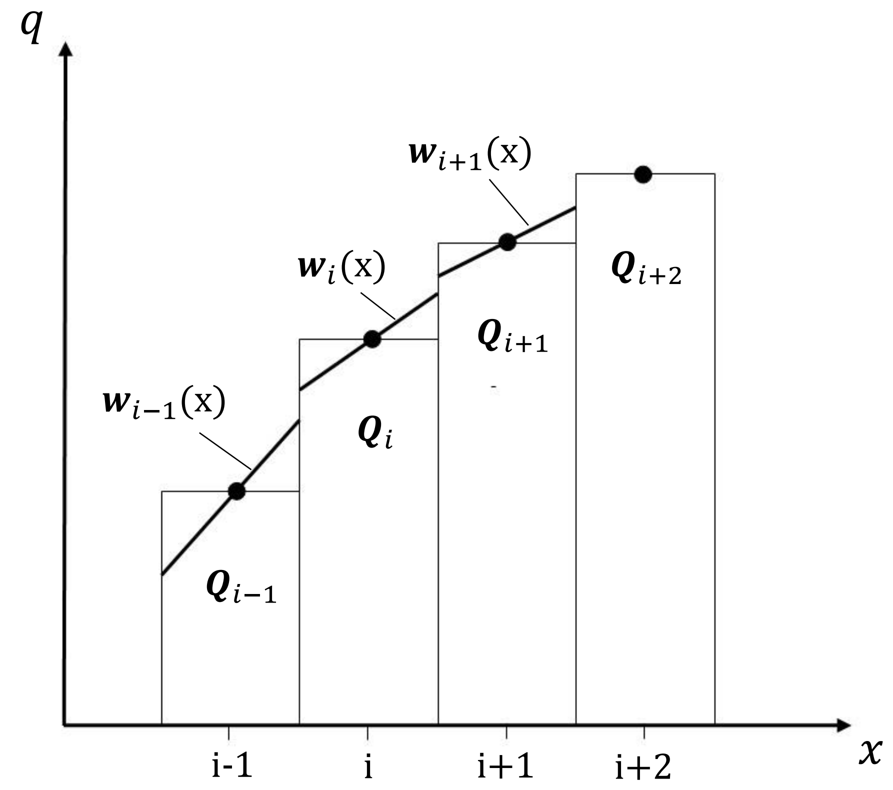
$$\boldsymbol{w}_i(\boldsymbol{x}, t^n) = (x - x_i)\boldsymbol{a}_i + (y - y_i)\boldsymbol{b}_i + \boldsymbol{Q}_i$$

Numerics

1) 1^{st} order PRICE scheme

2) Reconstruction in space

3) Slope limiting



$$\widehat{\boldsymbol{w}}_i(\boldsymbol{x}, t^n) = (x - x_i)\widehat{\boldsymbol{a}}_i + (y - y_i)\widehat{\boldsymbol{b}}_i + \boldsymbol{Q}_i$$

Numerics

1) 1^{st} order PRICE scheme

2) Reconstruction in space

3) Slope limiting

4) Reconstruction in time

$$\begin{aligned}
\boldsymbol{Q}_i(\boldsymbol{x}, t) &= \boldsymbol{Q}_i^n + (x - x_G)\frac{\partial \boldsymbol{Q}}{\partial x} + (y - y_G)\frac{\partial \boldsymbol{Q}}{\partial y} \\
&\quad + (t - t^n)\frac{\partial \boldsymbol{Q}}{\partial t} \\
&= \boldsymbol{Q}_i^n + (x - x_G)\frac{\partial \boldsymbol{Q}}{\partial x} + (y - y_G)\frac{\partial \boldsymbol{Q}}{\partial y} \\
&\quad -(t - t^n)\left(\boldsymbol{A}_x(\boldsymbol{Q}_i^n)\frac{\partial \boldsymbol{Q}}{\partial x} + \boldsymbol{A}_y(\boldsymbol{Q}_i^n)\frac{\partial \boldsymbol{Q}}{\partial y}\right)
\end{aligned}$$

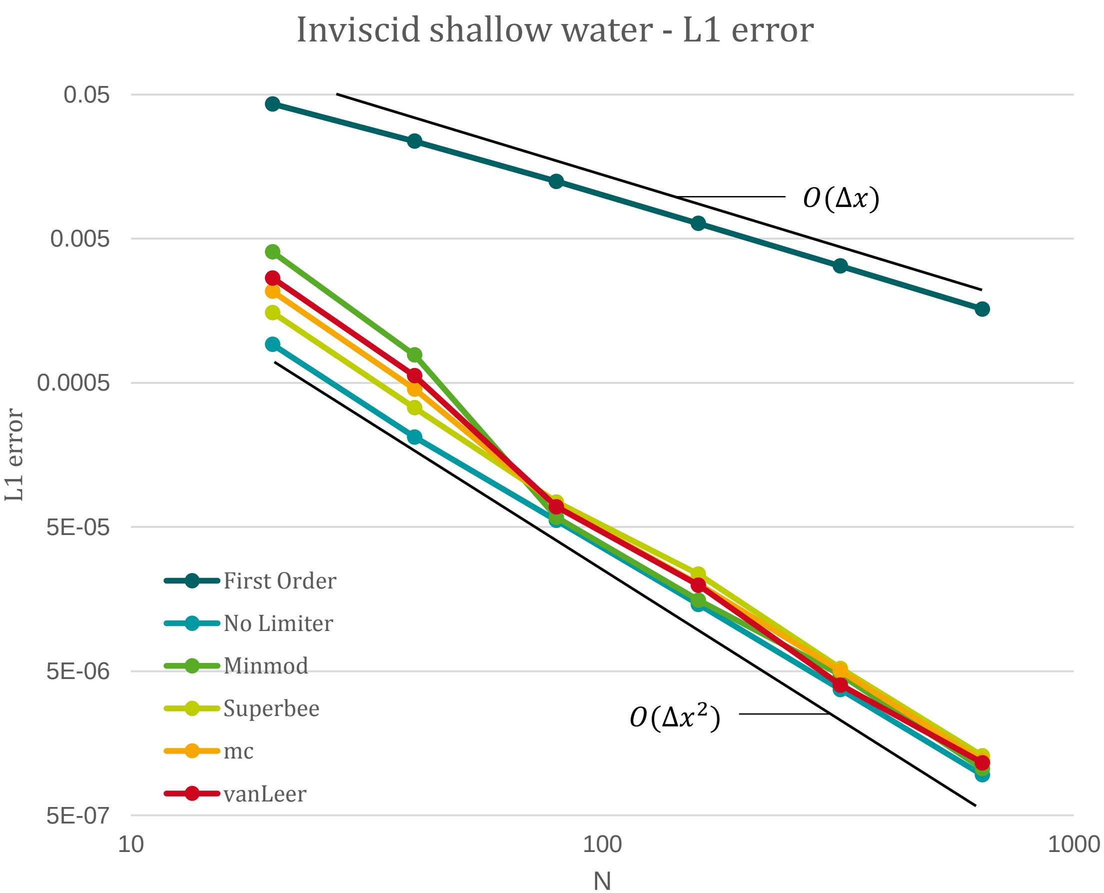
Convergence study - shallow water

- Inviscid shallow water equations

$$Q = \begin{pmatrix} h \\ q_x \\ q_y \\ b \end{pmatrix} \quad A_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{q_x^2}{h^2} + gh & \frac{2q_x}{h} & 0 & gh \\ -\frac{q_x q_y}{h^2} & \frac{q_y}{h} & \frac{q_x}{h} & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

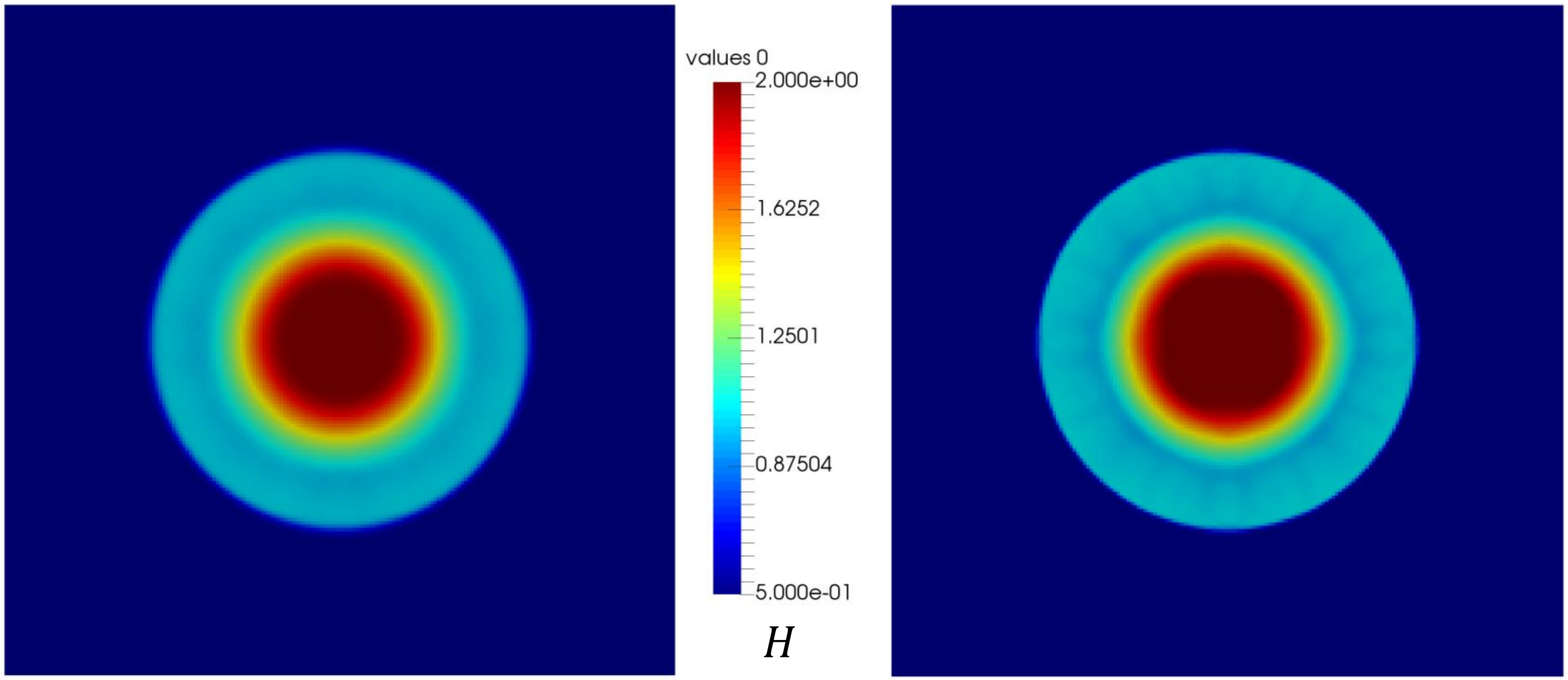
$$A_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\frac{q_x q_y}{h^2} & \frac{q_y}{h} & \frac{q_x}{h} & 0 \\ -\frac{q_y^2}{h^2} + gh & 0 & \frac{2q_y}{h} & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Non-linear, non-conservative
- Smooth test with known solution
- Fixed CFL number



Dam break problem

- Shallow water equations with fixed bed
- Initial discontinuity, $T_{end} = 1s$



First order

Second order (minmod)

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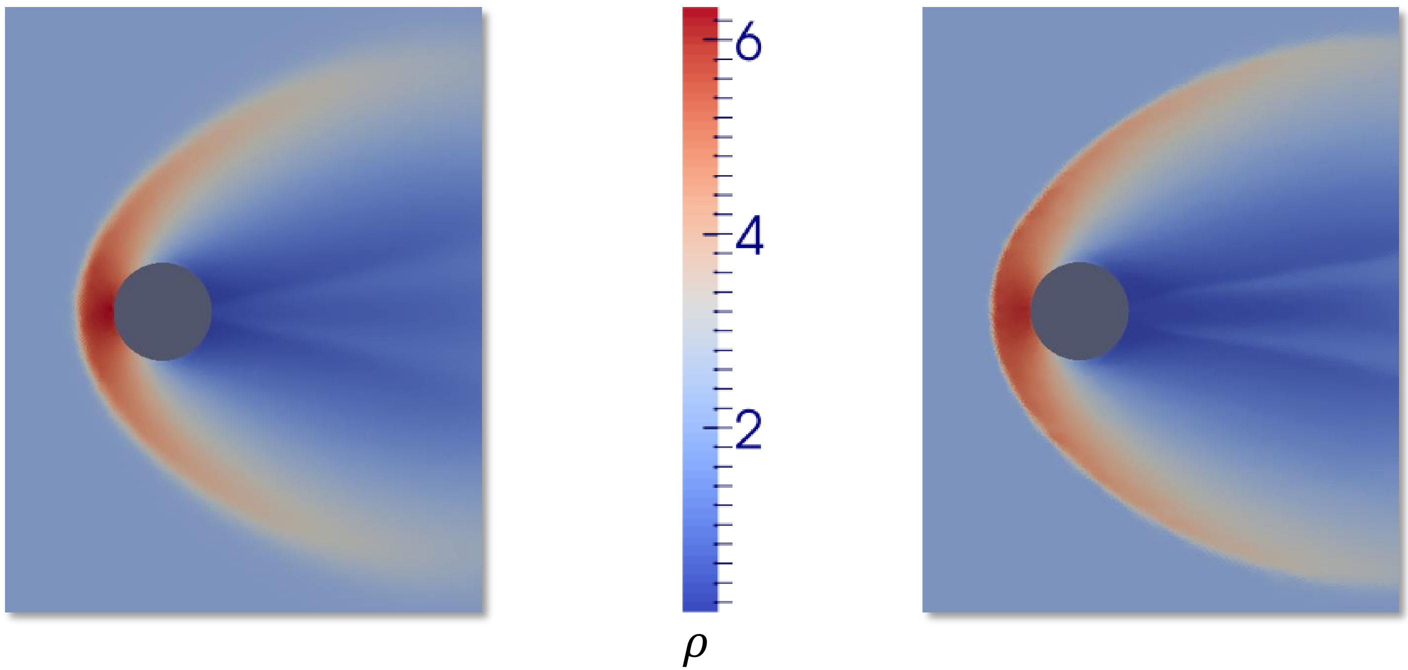
Flow past sphere

- Extended fluid dynamics equations for $Kn = 0.0005$
- $Ma = 3$, slip boundary conditions, until $T = 3s$

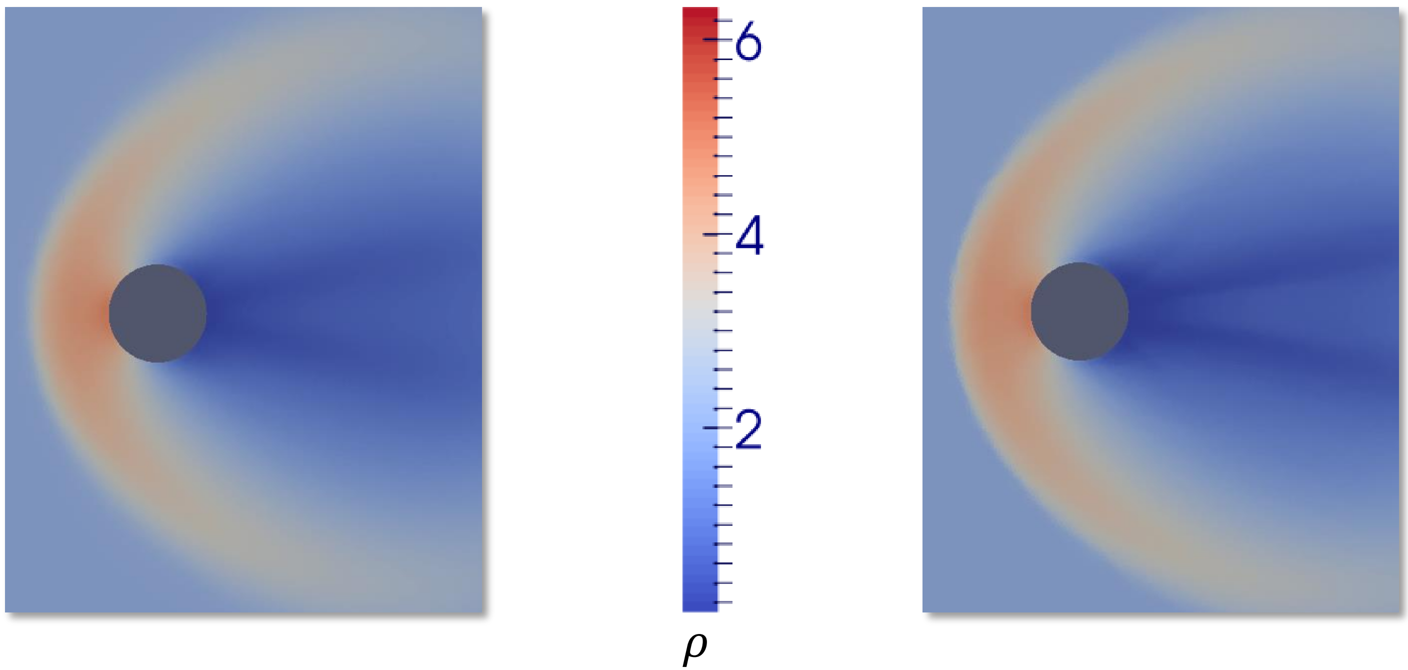


First order

Second order (minmod)

Rarefied flow past sphere

- Extended fluid dynamics equations for $Kn = 0.5$
- $Ma = 3$, slip boundary conditions, until $T = 3s$



First order

Second order (minmod)

Conclusion

- 2^{nd} order scheme for non-conservative PDEs on 2D unstructured grids
- Convergence test and improved solution quality
- Application for rarefied gas flows

Future work: Investigation of models for rarefied gas flows

Thank you for your attention

References

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