On a New Diagram Notation for the Derivation of Hyperbolic Moment Models

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Diagram-Based Derivation of Hyperbolic Moment Models

- Germany’s largest technical university
- 45k students, 10k staff
Introduction

Aim

Derive hyperbolic PDE systems for rarefied gas flows

Extension of standard fluid dynamic equations

- Reentry flows
- Micro channel flows
Introduction

Aim

Derive hyperbolic PDE systems for rarefied gas flows

Extension of standard fluid dynamic equations

- Reentry flows
- Micro channel flows

Importance of Hyperbolicity

- Physical solutions with bounded propagation speeds
- Well-posedness and stability of the solution
**Goal**

solve and simulate flow problems involving rarefied gases

**Knudsen number**

distinguish flow regimes by orders of the Knudsen number

\[ Kn = \frac{\text{mean free path length}}{\text{reference length}} = \frac{\lambda}{L} \]
Boltzmann Transport Equation

\[
\frac{\partial}{\partial t} f(t, x, c) + c_i \frac{\partial}{\partial x_i} f(t, x, c) = S(f)
\]

PDE for particles' *probability density function* \( f(t, x, c) \)

- Describes change of \( f \) due to transport and collisions
- Collision operator \( S \)
- Usually a 7-dimensional phase space
Boltzmann Transport Equation 1D

\[ \frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = S(f) \]

PDE for particles’ *probability density function* \( f(t, x, c) \)

- Describes change of \( f \) due to transport and collisions
- Collision operator \( S \)
- 3-dimensional phase space
Macroscopic Quantities

\( f(t, x, c) \) is related to the macroscopic quantities

density \( \rho(t, x) \), velocity \( v(t, x) \), temperature \( \theta(t, x) \)

\[
\begin{align*}
\rho(t, x) &= \int_{\mathbb{R}} f(t, x, c) \, dc \\
\rho(t, x)v(t, x) &= \int_{\mathbb{R}} cf(t, x, c) \, dc \\
\frac{1}{2} \rho(t, x)\theta(t, x) + \frac{1}{2} \rho(t, x)v(t, x)^2 &= \int_{\mathbb{R}} \frac{1}{2} c^2 f(t, x, c) \, dc
\end{align*}
\]
Review of Hyperbolic Moment Models
A Transformed Velocity Variable

\[
f(c) = \frac{\rho}{\sqrt{2\pi\theta}} \exp\left(-\frac{(c - v)^2}{2\theta}\right)
\]
A Transformed Velocity Variable

\[ \xi(t, x, c) := \frac{c - v(t, x)}{\sqrt{\theta(t, x)}} \]

\[ f(c) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left( -\frac{(c - v)^2}{2\theta} \right) \]
A Transformed Velocity Variable

\[ \xi(t, x, c) := \frac{c - v(t, x)}{\sqrt{\theta(t, x)}} \]

\[ f(c) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left(-\frac{(c - v)^2}{2\theta} \right) \]

\[ f(\xi) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left(-\frac{\xi^2}{2} \right) \]
A Transformed Velocity Variable

\[ \xi(t, x, c) := \frac{c - v(t, x)}{\sqrt{\theta(t, x)}} \]

\[ f(c) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left( -\frac{(c - v)^2}{2\theta} \right) \]

\[ f(\xi) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left( -\frac{\xi^2}{2} \right) \]

Transformed velocity space reduces numerical complexity
Model Order Reduction

Ansatz: Expansion

\[ f(t, x, c) = \sum_{\alpha=0}^{M} f_\alpha(t, x) \phi_{\alpha}^{v, \theta}(\xi) \]

Basis function: is weighted Hermite polynomial

\[ \phi_{\alpha}^{v, \theta}(\xi) = \frac{1}{\sqrt{2\pi}} \theta^{-\frac{\alpha+1}{2}} \exp \left( -\frac{\xi^2}{2} \right) \mathcal{H}_\alpha(\xi) \]
Model Order Reduction

**Ansatz: Expansion**

\[ f(t, x, c) = \sum_{\alpha=0}^{M} f_\alpha(t, x) \phi^{v, \theta}_\alpha(\xi) \]

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\[ \phi^{v, \theta}_\alpha(\xi) = \frac{1}{\sqrt{2\pi}} \theta^{-\frac{\alpha+1}{2}} \exp \left( -\frac{\xi^2}{2} \right) \mathcal{H}_\alpha(\xi) \]

**Reduction of Complexity**

One PDE for \( f(t, x, c) \) that is high-dimensional

\[ \Downarrow \]

System of PDEs for \( \rho(t, x), v(t, x), \theta(t, x), f_\alpha(t, x) \) that is low-dimensional
**Grad’s Method [Grad, 1949]**

### Galerkin Approach

- Standard method
- Multiplication with test function and integration

### Grad result

\[
\partial_t \mathbf{u}_M + \mathbf{A}_{\text{Grad}} \partial_x \mathbf{u}_M = 0, \\
\mathbf{u}_4 = (\rho, v, \theta, f_3, f_4)^T
\]

\[
\mathbf{A}_{\text{Grad}} = \begin{pmatrix}
v & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & v & 1 & 0 & 0 \\
0 & 2\theta & v & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v
\end{pmatrix}
\]

\[\Rightarrow \text{Loss of hyperbolicity}\]
Hyperbolic Moment Equations (HME) [CAI et al., 2012]

Modification of equations
- Based on Grad’s method
- Modification of last equation to achieve hyperbolicity

HME result
\[
\partial_t \mathbf{u}_M + \mathbf{A}_{\text{HME}} \partial_x \mathbf{u}_M = 0,
\]
\[
\mathbf{u}_4 = (\rho, v, \theta, f_3, f_4)^T
\]
\[
\mathbf{A}_{\text{HME}} = \begin{pmatrix}
n & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & v & 1 & 0 & 0 \\
0 & 2\theta & v & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & v
\end{pmatrix}
\]

\[\Rightarrow \text{Globally hyperbolic for every state vector } \mathbf{u}_M\]
Quadrature-Based Projection Approach

- Based on Grad’s method
- Substitution of integrals by Gaussian quadrature

QBME result

\[ \partial_t u_M + A_{QBME} \partial_x u_M = 0, \]

\[ A_{QBME} = \begin{pmatrix}
  v & \rho & 0 & 0 & 0 \\
  \frac{\theta}{\rho} & v & 1 & 0 & 0 \\
  0 & 2\theta & v & \frac{6}{\rho} & 0 \\
  0 & 4f_3 & \frac{\rho\theta}{2} & -\frac{10f_4}{\theta} & v \\
 -\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho\theta} & v
\end{pmatrix} \]

\[ u_4 = (\rho, v, \theta, f_3, f_4)^T \]

\[ u_4 = \begin{pmatrix}
  \rho \\
  v \\
  \theta \\
  f_3 \\
  f_4
\end{pmatrix} \]

\[ \Rightarrow \text{Globally hyperbolic for every state vector } u_M \]
Additions and Developments

Hyperbolic Moment Models

- Extension to $n$-D
- Framework for hyperbolic models
- ...

Applications

- Relativistic Boltzmann equation
- Quantum Gas
- ...

Hyperbolic Moment Models

- Extension to $n$-D
- Framework for hyperbolic models
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Applications

- Relativistic Boltzmann equation
- Quantum Gas
- ...
Diagram-Based Derivation
Derivation of Moment System

\[ f(t, x, c) = \sum_{\alpha=0}^{M} f_{\alpha}(t, x) \phi_{\alpha}^{v, \theta}(\xi) \rightarrow \frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = 0 \]

\[ \downarrow \]

\[ \left( \frac{\partial f_{\alpha}}{\partial t} + v \frac{\partial f_{\alpha}}{\partial x} + \theta \frac{\partial f_{\alpha-1}}{\partial x} + (\alpha + 1) \frac{\partial f_{\alpha+1}}{\partial x} + f_{\alpha-1} \frac{\partial v}{\partial t} + (vf_{\alpha-1} + \theta f_{\alpha-2} + (\alpha + 1)f_{\alpha}) \frac{\partial v}{\partial x} + f_{\alpha-2} \frac{\partial \theta}{\partial t} + \frac{1}{2} (vf_{\alpha-2} + \theta f_{\alpha-3} + (\alpha + 1)f_{\alpha-1}) \frac{\partial \theta}{\partial x} \right) \phi_{\alpha}^{v, \theta}(\xi) = 0 \]
Derivation of Moment System (2)

\[ \partial_t (\cdot) + c \partial_x (\cdot) \]
Towards a Diagram-Based Derivation

Previous derivations
- Concise and mathematically elegant way to derive models
- Very theoretical and no direct insight into equations

Diagram-Based Notation
- Better understanding of hyperbolic regularization
- Derivation of new models
Derivation 1

Equation and ansatz

\[ \partial_t f(t, x, c) + c \cdot \partial_x f(t, x, c) = 0 \]

\[ f(t, x, c) = \sum_{\alpha=0}^{M} f_\alpha(t, x) \phi_{\alpha}^{v, \theta}(\xi), \quad \xi = \frac{c - v(t, x)}{\sqrt{\theta(t, x)}} \]

\[ \partial_s f_\alpha \phi_\alpha(\xi) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_s \phi_\alpha(\xi) \]

\[ = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \left( \partial_\theta \phi_\alpha(\xi) \partial_s \theta + \partial_\xi \phi_\alpha(\xi) \partial_s \xi \right) \]

\[ = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \left( \partial_\theta \phi_\alpha(\xi) \partial_s \theta + \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \right) \]
Equation and ansatz

\[ \partial_t f(t, x, c) + c \cdot \partial_x f(t, x, c) = 0 \]

\[ f(t, x, c) = \sum_{\alpha=0}^{M} f_{\alpha}(t, x) \phi_{\alpha}^{v, \theta}(\xi), \quad \xi = \frac{c - v(t, x)}{\sqrt{\theta(t, x)}} \]

\[ \partial_s (f_{\alpha} \phi_{\alpha}(\xi)) = \partial_s f_{\alpha} \phi_{\alpha}(\xi) + f_{\alpha} \left( \partial_\theta \phi_{\alpha}(\xi) \partial_s \theta + \partial_\xi \phi_{\alpha}(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \right) \]

\[ c \cdot f_{\alpha} \phi_{\alpha}(\xi) = \left( v + \sqrt{\theta} \xi \right) \cdot f_{\alpha} \phi_{\alpha}(\xi) \]
Choice of Basis Function

weighted Hermite polynomial

\[ \phi_\alpha(\xi) = \frac{1}{\sqrt{2\pi}} \theta^{-\frac{\alpha+1}{2}} H_\alpha(\xi) \exp\left(-\frac{\xi^2}{2}\right) \]

1. \( \xi \) derivative:
   \[ \frac{\partial}{\partial \xi} \phi_\alpha(\xi) = -\sqrt{\theta} \phi_{\alpha+1}(\xi) \]

2. \( \theta \) derivative:
   \[ \frac{\partial}{\partial \theta} \phi_\alpha(\xi) = -\frac{\alpha+1}{2\theta} \phi_\alpha(\xi) \]

3. multiplication with \( \xi \):
   \[ \xi \phi_\alpha(\xi) = \sqrt{\theta} \phi_{\alpha+1}(\xi) + \frac{\alpha}{\sqrt{\theta}} \phi_{\alpha-1}(\xi) \]
\[ \frac{\partial}{\partial \xi} \phi_{\alpha}(\xi) = -\sqrt{\theta} \phi_{\alpha+1}(\xi) \]
$\frac{\partial}{\partial \theta} \phi_\alpha(\xi) = -\frac{\alpha + 1}{2\theta} \phi_\alpha(\xi)$
\[ \xi \phi_\alpha (\xi) = \sqrt{\theta} \phi_{\alpha+1} (\xi) + \frac{\alpha}{\sqrt{\theta}} \phi_{\alpha-1} (\xi) \]
Diagram Notation

Aim

1. Extend diagram notation from basis function to whole equation
2. Split terms of equation into steps
3. Use single diagram for each step

\[
\begin{align*}
\partial_t f(t, x, c) + c \cdot \partial_x f(t, x, c) &= 0 \\
\partial_s (f_\alpha \phi_\alpha(\xi)) &= \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \\
2a &+ 2b + 1 \cdot 2 \\
c \cdot f_\alpha \phi_\alpha(\xi) &= \left( v + \sqrt{\theta} \xi \right) \cdot f_\alpha \phi_\alpha(\xi) \\
3
\end{align*}
\]
Diagram-Based Derivation of Hyperbolic Moment Models

\[ f_\alpha \cdot \partial_\xi \phi_\alpha(\xi) = f_\alpha \cdot (-\sqrt{\theta}) \cdot \phi_{\alpha+1}(\xi) \]
\[ f_\alpha \cdot \phi_\alpha \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) = f_\alpha \cdot \left( -\frac{\alpha \partial_s \theta}{2\sqrt{\theta}} \cdot \phi_{\alpha-1} - \frac{\partial_s v}{\sqrt{\theta}} \cdot \phi_\alpha - \frac{\partial_s \theta}{2\sqrt{\theta}} \cdot \phi_{\alpha+1} \right) \]
\[ (\partial_s f_\alpha) \phi_\alpha(\xi) = \partial_s f_\alpha \cdot \phi_\alpha(\xi) \]
\[ f_{\alpha} \partial_\theta \phi_\alpha(\xi) \partial_s \theta = f_{\alpha} \cdot \left( -\frac{\alpha + 1}{2\theta} \partial_s \theta \right) \cdot \partial_s \phi_\alpha(\xi) \]
\[
(v + \sqrt{\theta}\xi) \cdot f_\alpha \phi_\alpha(\xi) = f_\alpha \cdot (\theta \phi_{\alpha-1}(\xi) + v \phi_\alpha(\xi) + \alpha \phi_{\alpha+1}(\xi))
\]
Grad diagram

\[
\partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right)
\]

\[
2a + 2b + 1 \cdot 2
\]
\[ \partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \]

\[ 2a + 2b + 1 \cdot 2 \]

\[ M - 2 \quad M - 1 \quad M \quad M + 1 \quad M + 2 \]

\[ \ldots \quad \bullet \quad \bullet \quad \bullet \quad \circ \quad \circ \quad \ldots \]

\[ \ldots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \ldots \]

\[ \ldots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \ldots \]

\[ P_1 \]
Grad diagram

\[ \partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \]

\[ 2a \quad + \quad 2b \quad + \quad 1 \cdot 2 \]

Diagram notation:

- \( M - 2 \)
- \( M - 1 \)
- \( M \)
- \( M + 1 \)
- \( M + 2 \)

\[ P_1 \]

\[ P_2 \]
\[
\partial_s (f_\alpha \phi_\alpha (\xi)) = \partial_s f_\alpha \phi_\alpha (\xi) + f_\alpha \partial_\theta \phi_\alpha (\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha (\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \\
\circ \quad + \quad 2b \quad + \quad 1 \quad \cdot \quad 2
\]
\[
\partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left(- \frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta\right)
\]

\[
2a + 2b + 1 \cdot 2
\]
Grad diagram

\[ \partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) \]

\[ 2a + 2b + 1 \cdot 2 \]
Single Basis Function

\[ f_{\alpha} \phi_{\alpha} \rightarrow \left( -\frac{\alpha + 1}{2\theta} \partial_x \theta \right) u f_{\alpha} \phi_{\alpha}, \]

\[ f_{\alpha} \phi_{\alpha} \rightarrow \left( -\sqrt{\theta} \right) \left( -\frac{\alpha + 1}{2\sqrt{\theta}^3} \partial_x \theta \right) u f_{\alpha} \phi_{\alpha} \]
Grad diagram

\[ M - 2 \quad M - 1 \quad M \quad M + 1 \quad M + 2 \]

... \[ P_1 \quad P_{2a} \quad P_{2b} \]

... \[ P_2 \]

... \[ P_3 \]

...
Diagram-Based Derivation of Hyperbolic Moment Models
Simplified Hyperbolic Moment Equations (SHME)

- QBME and HME cut off terms during derivation
- Corresponds to new formulas, different adaptivity

⇒ How much adaptivity is necessary?

\[
\partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left( -\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right) 
\]

\[
\approx \partial_s f_\alpha \phi_\alpha(\xi)
\]

\[
c \cdot f_\alpha \phi_\alpha(\xi) = \left( v + \sqrt{\theta} \xi \right) \cdot f_\alpha \phi_\alpha(\xi)
\]

- Neglect adaptivity of \( \partial_s (\cdot) \)
- Keep adaptivity of \( c \cdot (\cdot) \)
\begin{equation}
\partial_s (f_\alpha \phi_\alpha(\xi)) = \partial_s f_\alpha \phi_\alpha(\xi) + f_\alpha \partial_\theta \phi_\alpha(\xi) \partial_s \theta + f_\alpha \partial_\xi \phi_\alpha(\xi) \left(-\frac{1}{\sqrt{\theta}} \partial_s v - \frac{\xi}{2\theta} \partial_s \theta \right)
\end{equation}
Simplified Hyperbolic Moment Equations (SHME)

Simplified derivation

- Neglecting parts of adaptivity

SHME result

\[ \partial_t \mathbf{u}_M + \mathbf{A}_{\text{SHME}} \partial_x \mathbf{u}_M = 0, \]

\[ \mathbf{u}_4 = (\rho, \nu, \theta, f_3, f_4)^T \]

\[ \mathbf{A}_{\text{SHME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\theta & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 0 & \frac{\rho\theta}{2} & \nu & 4 \\
0 & 0 & 0 & \theta & \nu
\end{pmatrix} \]

\[ \Rightarrow \text{Globally hyperbolic (equilibrium)} \]
Numerical Results
Conservative PDE systems

Standard conservative PDE system

$$\partial_t u + \partial_x F(u) = 0$$
Conservative PDE systems

Standard conservative PDE system

\[ \partial_t u + \partial_x F(u) = 0 \]

Basic Finite Volume scheme

\[ u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( F^*_{i+\frac{1}{2}} - F^*_{i-\frac{1}{2}} \right) \]

- Numerical flux \( F^*_{i+\frac{1}{2}} \) needed
- Conservation property by design
Non-conservative PDE systems

\[ \partial_t u + A(u) \partial_x u = 0 \]
Non-conservative PDE systems

Non-conservative PDE system

\[ \partial_t u + A(u) \partial_x u = 0 \]

Standard conservative PDE system

\[ \partial_t u + \partial_x F(u) = 0 \]

- Can be written in conservative form iff \( A(u) = \frac{\partial F(u)}{\partial u} \)
- In general no flux function available
- What about partially conservative systems?

⇒ Special numerical methods are needed
Numerical Methods

Wave Propagation scheme [LeVeque, 1997]
- 2nd order upwind type scheme

Castro scheme [Castro, Pares, 2004]
- 1st order upwind type scheme

PRICE-C scheme [Canestrelli, 2009]
- 1st order centered scheme
Riemann problem with BGK collision operator

\[
\frac{\partial}{\partial t} \mathbf{w} + \mathbf{A}(\mathbf{w}) \frac{\partial}{\partial x} \mathbf{w} = -\frac{1}{\tau} \mathbf{P} \mathbf{w}, \quad x \in [-2, 2]
\]

- \[ \rho_L = 7, \rho_R = 1 \]

- Variable vector \( \mathbf{w} = (\rho, u, \theta, f_3, f_4) \)
- Relaxation time \( \tau = \frac{Kn}{\rho} \Rightarrow \text{non-linear} \)
Model Equations

Grad model

\[
\mathbf{A}_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]

HME model

\[
\mathbf{A}_{\text{HME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & \nu
\end{pmatrix}
\]
Model Equations 2

**Grad model**

\[
\mathbf{A}_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]

**QBME model**

\[
\mathbf{A}_{\text{QBME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & -\frac{10f_4}{\theta} & \nu & 4 \\
-\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15f_4}{\rho\theta} & \nu
\end{pmatrix}
\]
### Model Equations 3

#### Grad model

\[
A_{\text{Grad}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 4f_3 & \frac{\rho\theta}{2} & 2f_3 & \nu & 4 \\
-f_3\frac{\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & \nu
\end{pmatrix}
\]

#### SHME model

\[
A_{\text{SHME}} = \begin{pmatrix}
\nu & \rho & 0 & 0 & 0 \\
\frac{\theta}{\rho} & \nu & 1 & 0 & 0 \\
0 & 2\theta & \nu & \frac{6}{\rho} & 0 \\
0 & 0 & \frac{\rho\theta}{2} & 4 & \nu \\
0 & 0 & 0 & \theta & \nu
\end{pmatrix}
\]
Method Comparison, $Kn = 0.05$

![Graph showing comparison of different methods]

- Blue line: PRICE
- Green line: UPRICE
- Red line: Castro
- Gray line: WP2

Parameters:
- $ho$: Density
- $u$: Velocity
Method Comparison, $Kn = 0.5$
Model Comparison, $Kn = 0.05$

![Diagram showing model comparison with different moment models for $Kn = 0.05$. The x-axis represents space (x), and the y-axis represents density ($\rho$) and velocity ($u$). The graph shows the behavior of different models (HME, SHME, DVM, QBME) at various space locations.]
Model Comparison, $Kn = 0.5$
Averaged Solution, $Kn = 0.5$

![Graph showing averaged solution for $Kn = 0.5$ with plots for different models: QBME8, QBME9, QBME8+9, and DVM. The graph displays variations in density ($\rho$), pressure ($p$), and velocity ($u$) across different regions.](image-url)
Model Convergence, $Kn = 0.5$

Figure: HME and QBME

Figure: SHME
2D bow shock, $Ma = 3$, $Kn = 0.0005$

Figure: 1st order

Figure: 2nd order
2D bow shock, $Ma = 3, Kn = 0.5$

**Figure:** 1st order

**Figure:** 2nd order
Hyperbolic Moment Equations

- Hierarchical models for rarefied gases
- Different models are available

Diagram notation

- Reveal details of derivation
- Allow derivation of new models

Further work

- M-adaptivity
Summary and Further Work

Hyperbolic Moment Equations
- Hierarchical models for rarefied gases
- Different models are available

Diagram notation
- Reveal details of derivation
- Allow derivation of new models

Further work
- M-adaptivity

Thank you for your attention
References

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