On new hyperbolic moment models for the Boltzmann equation

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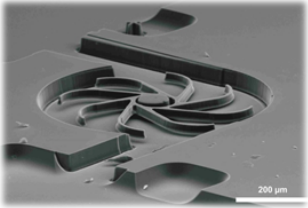
Introduction

Aim

Derive hyperbolic PDE systems for rarefied gas flows

Extension of standard fluid dynamic equations

- Reentry flows
- Micro channel flows



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Importance of Hyperbolicity

- Well-posedness and stability of the solution

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

PDE for particles' *probability density function* $f(t, \mathbf{x}, \mathbf{c})$

- Describes change of f due to transport and collisions
- Collision operator S
- Usually a 7-dimensional phase space

Model Order Reduction

Ansatz

$$f(t,\mathbf{x},\mathbf{c}) = \sum_{i=0}^{M} f_i(t,\mathbf{x})\mathcal{H}_i^{\rho,\mathbf{v},\theta}(\mathbf{c})$$

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One PDE for $f(t,\mathbf{x},\mathbf{c})$ that is 7-dimensional

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$$\Downarrow$$

System of PDEs for $\rho(t,\mathbf{x}), \mathbf{v}(t,\mathbf{x}), \theta(t,\mathbf{x}), f_i(t,\mathbf{x})$ that is 4-dimensional

Grad's Method [Grad, 1949]

Galerkin Approach

- Standard method
- Multiplication with test function and integration

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Grad result

$$\partial_t \mathbf{u}_M + \mathbf{A}_{\mathrm{Grad}} \partial_x \mathbf{u}_M = 0,$$

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$\Rightarrow$ Only locally hyperbolic

Quadrature-Based Moment Equations (QBME) [2014]

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$\Rightarrow$ Globally hyperbolic for every state vector $\mathbf{u}_M$

Numerical Methods

Non-conservative PDE system

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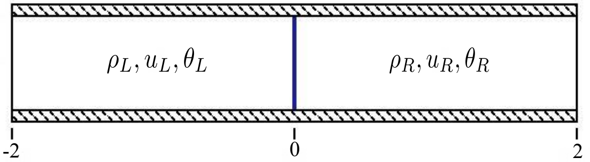
PRICE-C scheme [Canestrelli, 2009]

- First order
- Centered scheme
- Implemented on 2D unstructured grids

Shock Tube Test Case



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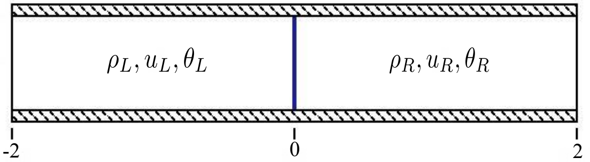


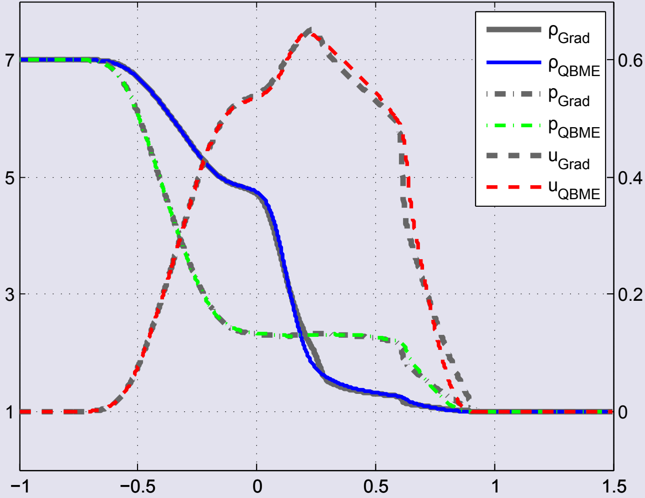
Riemann problem with BGK collision operator

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = -\frac{1}{\tau}\mathbf{P}\mathbf{u}, \quad x \in [-2, 2]$$

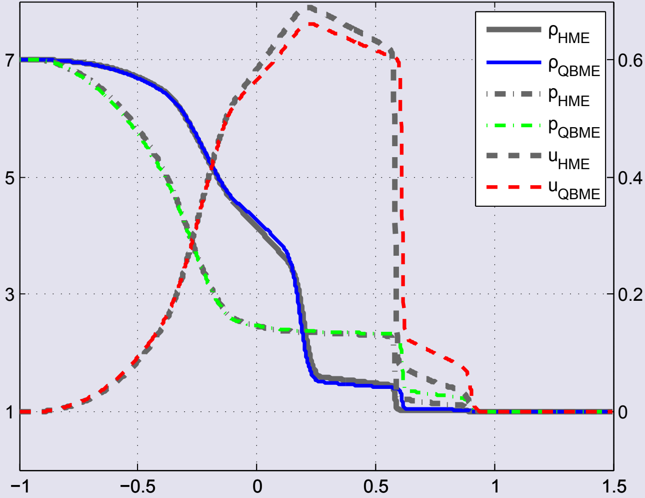
$$\rho_L = 7, \rho_R = 1$$

- Variable vector $\mathbf{u} = (\rho, u, \theta, f_3, f_4)$
- Relaxation time $\tau = \frac{\mathrm{Kn}}{\rho} \Rightarrow$ non-linear

$\mathrm{Kn} = 0.05$ and comparison with Grad



$\mathrm{Kn} = 0.5$ and comparison with HME



Summary and Further Work

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- Global hyperbolicity
- Easy extension to multi-dimensional case
- First simulation results show good agreement with existing models

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Thank you for your attention

References

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Hyperbolic Moment Equations (HME) [Cai et al., 2014]

Modification of equations

- Based on Grad's method
- Modify terms by projections to achieve hyperbolicity

HME result

$$\partial_t \mathbf{u}_M + \mathbf{A}_{\text{HME}} \partial_x \mathbf{u}_M = 0,$$

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