On Derivation and Analysis of Moment Models for the Shallow Water Equation

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Today’s Topic

- Quadrature-Based Moment Equations
- Filtered Hyperbolic Moment Equations
- Projective Integration for Moment Equations
- Hyperbolic Shallow Water Moment Equations
- Adaptive Moment Model
Today’s Topic

- Quadrature-Based Moment Equations
- Filtered Hyperbolic Moment Equations
- Projective Integration for Moment Equations
- **Hyperbolic Shallow Water Moment Equations**
- Adaptive Moment Model
Shallow Water Moment Equations
1 Shallow Water Moment Equations
   - Classical Shallow Water Equations
   - Transformation of z-axis
   - Moment method for SWE

2 Hyperbolic Regularization of SWME
   - Hyperbolicity of SWME
   - Linear regularization
   - $\beta$-regularization

3 Numerical results

4 Summary
Starting point: Incompressible Navier-Stokes equations

Mass: $\nabla \cdot \mathbf{u} = 0$

Momentum: $\partial_t \mathbf{u} + \nabla \cdot (\mathbf{uu}) = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \sigma + \mathbf{g}$

with velocity vector $\mathbf{u} = (u, v, w)^T$. 

Water surface $h_s$ and bottom topography $h_b$

Water height $h = h_s - h_b$.

Hydrostatic pressure: $p(x, y, t, z) = (h_s(t, x, y) - z)\rho g e_z$
Assumption

Velocity profile is constant over height

Velocity components are averaged over height:

$$\bar{u}(t, x, y) = \int_{h_b}^{h_s} u \, dz \quad , \quad \bar{v}(t, x, y) = \int_{h_b}^{h_s} v \, dz$$
Classical Shallow Water Equations (1D)

\[ \partial_t h + \partial_x (h\bar{u}) = 0 \]

\[ \partial_t (h\bar{u}) + \partial_x \left( h\bar{u}^2 + \frac{1}{2}gh^2 \right) = 0 . \]

kinematic boundary conditions

\[ \partial_t h_s + \begin{pmatrix} u \\ v \end{pmatrix} \cdot \nabla h_s = w \quad \text{for } z = h_s , \]

\[ \partial_t h_b + \begin{pmatrix} u \\ v \end{pmatrix} \cdot \nabla h_b = w \quad \text{for } z = h_b \]
Transformation of z-axis

\[ z \mapsto \zeta = z - h_b \]

We consider the transformed functions, e.g., transformed pressure
\[ p \mapsto \tilde{p}(x, y, t, \zeta) = h(1 - \zeta) \rho g \]
Transformation of $z$-axis

$$z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h}$$
Transformation of z-axis

\[ z \mapsto \zeta = \frac{z - h_b}{h_s - h_b} = \frac{z - h_b}{h} \]

We consider the transformed functions, e.g. transformed pressure

\[ p \mapsto \tilde{p}(x, y, t, \zeta) = h(1 - \zeta) \rho g \]
Transformed conservation of mass

\[ h \partial_x u + h \partial_y v + h \partial_z w = 0 \]

1. insert transformation

\[ h \partial_s \psi = \partial_s \left( h \bar{\psi} \right) - \partial_\zeta \left( \partial_s (\zeta h + h_b) \bar{\psi} \right) \]

2. integrate equation

3. use kinematic boundary conditions

\[ \partial_t h + \partial_x (hu_m) + \partial_y (hv_m) = 0 \]
Transformed conservation of momentum

- transform pressure
  
  \[ \tilde{p}(t, x, y, \zeta) := p(t, x, y, \zeta h + h_b) = h(1 - \zeta) \rho g . \]

- transform pressure derivative
  
  \[ \frac{1}{\rho} \partial_x p = \partial_x \left( \frac{g}{2} h^2 \right) + hg \partial_x h_b \]

- replace z-velocity \( \tilde{w}(t, x, y, \zeta) \) using integrated conservation of mass
  
  \[ \tilde{w}(t, x, y, \zeta) = -\partial_x \left( h \int_0^\zeta \tilde{u} \, d\tilde{\zeta} \right) - \partial_y \left( h \int_0^\zeta \tilde{v} \, d\tilde{\zeta} \right) \]
  
  \[ + \partial_x (\zeta h + h_b) \tilde{u} + \partial_y (\zeta h + h_b) \tilde{v} + \partial_t h_b . \]
Transformed conservation of momentum (2)

- use coupling parameter $\omega$

$$ h\omega(h, \tilde{u}, \tilde{v}) := \partial_x \left( h \int_0^\zeta u_m - \tilde{u} \, d\tilde{\zeta} \right) + \partial_y \left( h \int_0^\zeta v_m - \tilde{v} \, d\tilde{\zeta} \right) $$

- coupling parameter $\omega$ does not depend on the vertical velocity, but describes the relation of all velocity components
Transformed shallow water equations

original SWE

\[ \partial_t h + \partial_x (h\bar{u}) = 0 \]
\[ \partial_t (h\bar{u}) + \partial_x \left( h\bar{u}^2 + \frac{1}{2}gh^2 \right) = -hg\partial_x h_b \]

transformed SWE

\[ \partial_t h + \partial_x (hu_m) = 0 \]
\[ \partial_t (hu) + \partial_x \left( hu^2 + \frac{g}{2}h^2 \right) + \partial_\zeta (hu\omega) = -hg\partial_x h_b \]
No slip boundary condition at the bottom,
Zero Neumann boundary condition at the top.

\[
\begin{align*}
\partial_\zeta u \mid_{\zeta=1} &= 0, \\
u \mid_{\zeta=0} &= 0,
\end{align*}
\]

Other boundary conditions possible, e.g. for slip boundary condition or friction at the bottom.
**Idea**

Allow for variations in $z$ direction

- multi-layer approach with many constant velocities
- continuous representation of velocity
Idea

Allow for variations in z direction

- multi-layer approach with many constant velocities
- continuous representation of velocity
Moment expansion for shallow water model

Idea from [Torrilhon, Kowalski, 2018]

"Moment Approximations and Model Cascades for Shallow Flow": Expand velocity in (scaled) z-direction around mean mean velocity

\[
 u(t, x, \zeta) = u_m(t, x) + \sum_{j=1}^{N} \alpha_j(t, x) \phi_j(\zeta)
\]

Mean of \( u \)

Deviation of \( u \)

\[
\phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2, \quad \phi_3(\zeta) = 1 - 12\zeta + 30\zeta^2 - 20\zeta^3
\]

Legendre polynomials \( \phi_j(\zeta) \) are orthogonal on the interval \([0, 1]\).

1. \( \langle \phi_i, \phi_j \rangle_{L^2[0,1]} = \frac{1}{2j+1} \delta_{ij} \), for \( i, j \in \{0, \ldots, N\} \)

2. \( \int_0^1 \phi_j \, d\zeta = 0 \) for \( j \in \{1, \ldots, N\} \)

Authors use \( \alpha_1 = s, \alpha_2 = \kappa, \alpha_3 = m \).
Moment Method for shallow water equations

Moment method

- Expand velocity in basis functions \( \phi_i \) and basis coefficients \( \alpha_i \)
- Insert expansion into transformed momentum equation
- Multiply with test function, e.g. \( \phi_j \) (orthogonal), \( i = 1, \ldots, N \)
- Integrate over vertical variable \( \zeta \)

Moment system

PDE for \( u \) and coefficients \( \alpha_i \)
transformed SWE

\[
\partial_t h + \partial_x(hu_m) = 0
\]

\[
\partial_t (hu) + \partial_x \left( hu^2 + \frac{g}{2} h^2 \right) + \partial_\zeta (hu\omega) = -hg \partial_x h_b
\]

moment model of transformed SWE

\[
\partial_t h + \partial_x(hu_m) = 0
\]

\[
\partial_t (hu_m) + \partial_x \left( hu_m^2 + \sum_{j=1}^{N} \frac{h \alpha_j^2}{2j+1} + \frac{g}{2} h^2 \right) = -hg \partial_x h_b, \quad (i = 0)
\]

\[
\partial_t (h\alpha_i) + \partial_x \left( 2hu_m + h \sum_{k,j=1}^{N} A_{ijk} \alpha_j \alpha_k \right) = u_m \partial_x (h\alpha_i) - \sum_{j,k=1}^{N} \alpha_k B_{ijk} \partial_x (h\alpha_j)
\]

\[
A_{ijk} = \int_0^1 \phi_i \phi_j \phi_k \ d\zeta, \quad B_{ijk} = (2i + 1) \int_0^1 \phi_i' \int_0^\zeta \phi_j \ d\tilde{\zeta} \phi_k \ d\zeta
\]
Zeroth order system (Shallow water equations):

\[ \partial_t \begin{pmatrix} h \\ hu_m \\ hs \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{gh^2}{2} \\ 2hu_ms \end{pmatrix} = -\frac{\nu}{\lambda} \begin{pmatrix} 0 \\ 0 \\ um\partial_x(hs) \end{pmatrix} \]

First order system:

\[ \partial_t \begin{pmatrix} h \\ hu_m \\ hs \end{pmatrix} + \partial_x \begin{pmatrix} hu_m^2 + \frac{gh^2}{2} + \frac{1}{3}hs^2 \\ 2hu_ms \\ 3(u_m+s+4\frac{\lambda}{h}s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ um\partial_x(hs) \end{pmatrix} -\frac{\nu}{\lambda} \begin{pmatrix} 0 \\ 0 \\ um + s + 4\frac{\lambda}{h}s \end{pmatrix} \]
Second order system:

$$\frac{\partial}{\partial t} \begin{pmatrix} h \\ hu_m \\ hs \\ h\kappa \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} hu_m \\ hu_m^2 + \frac{gh^2}{2} + \frac{1}{3} hs^2 + \frac{1}{5} h\kappa^2 \\ 2hu_ms + \frac{4}{5} hs\kappa \\ 2hu_m\kappa + \frac{2}{3} hs^2 + \frac{2}{7} h\kappa^2 \end{pmatrix} = Q \frac{\partial}{\partial x} \begin{pmatrix} h \\ hu_m \\ hs \\ h\kappa \end{pmatrix} - \frac{\nu}{\lambda} P$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\kappa}{5} & \frac{s}{5} \\ 0 & 0 & s & u_m + \frac{\kappa}{7} \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ u_m + s + \kappa \\ 3(u_m + s + \kappa + 4\frac{\lambda}{h} s) \\ 5(u_m + s + \kappa + 12\frac{\lambda}{h}\kappa) \end{pmatrix}$$

...
### Computation of eigenvalues
- Zeroth order and first order system: globally hyperbolic
- Higher order systems: not globally hyperbolic  
  Loss of hyperbolicity for small deviation from equilibrium!

### Breakdown of hyperbolicity
- no real propagation speeds
- problems for numerical method, nonphysical oscillations
Figure: Second order (left) and third order (right, for $\kappa = 0$)
Figure: Initial cubic velocity $u(0, x, \zeta)$ with coefficients $s_0 = -0.25$, $\kappa_0 = 0$ and $\gamma_0 = 0.26$
Breakdown of hyperbolicity

Figure: Hyperbolic breakdown (red) for $x_1 = -0.5$; $x_2 = 0$; $x_3 = 0.5$. 
Linear regularization

Idea

- Change system matrix to obtain hyperbolicity
- Preserve structure and conservation of mass

Linear regularization

- Linearization around \((h, u_m, s, 0, \ldots, 0)\)
- Hyperbolic beyond second order model
Linear regularization

Second order system

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
gh - u_m^2 - \frac{1}{3}s^2 - \frac{1}{5}\kappa^2 & 2u_m & \frac{2}{3}s & \frac{2}{5}\kappa \\
-2u_ms - \frac{4}{5}s\kappa & 2s & u_m + \kappa & \frac{3}{5}s \\
\frac{2}{3}s^2 - 2u_m\kappa - \frac{2}{7}\kappa^2 & 2\kappa & \frac{1}{3}s & u_m + \frac{3}{7}\kappa
\end{pmatrix}
\]

Idea

Linearize the system around \((h, u_m, s, 0, \ldots, 0)\)

\[
A_{\text{mod}} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
gh - u_m^2 - \frac{1}{3}s^2 & 2u_m & \frac{2}{3}s & 0 \\
-2u_ms & 2s & u_m & \frac{3}{5}s \\
\frac{2}{3}s^2 & 0 & \frac{1}{3}s & u_m
\end{pmatrix}
\]

Eigenvalues \(a_{1,2} = u_m \pm \sqrt{gh + s^2}\) and \(a_{3,4} = u_m \pm \frac{s}{\sqrt{5}}\).
Linear regularization

- Successful regularization for $N \leq 5$
- Loss of hyperbolicity for $N > 5$

**Idea**

- additionally change last equation using parameters $\beta_i$
- find parameters $\beta_i$ by matching characteristic polynomial with given polynomial
Linear regularization

- Successful regularization for $N \leq 5$
- Loss of hyperbolicity for $N > 5$

Idea
- additionally change last equation using parameters $\beta_i$
- find parameters $\beta_i$ by matching characteristic polynomial with given polynomial

$\Rightarrow \beta$-regularization
1. Linearize the system matrix around \((h, u_m, s, 0, \ldots, 0)\)
2. Add parameters \(\beta_i\) to last row of linearized system matrix
3. Compute characteristic polynomial \(\chi\)
4. Compute desired characteristic polynomial \(\chi_0\)
5. Solve for uniquely defined \(\beta_i\) by matching coefficients of the characteristic polynomials
Example for $\beta$-regularization: $N = 3$

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
gh - u_m^2 - \frac{1}{3}s^2 & 2u_m & \frac{2}{3}s & 0 & 0 \\
-2u_ms & 2s & u_m & \frac{3}{5}s & 0 \\
\frac{2}{3}s^2 & 0 & \frac{1}{3}s & u_m & \frac{4}{7}s \\
\beta_1 & \beta_2 & \beta_3 & \frac{2}{5}s + \beta_4 & u_m + \beta_5
\end{pmatrix}$$

Consider the propagation speeds

$$a_{1,2} = u_m \pm \sqrt{\frac{3}{5}} \sqrt{gh + s^2}, \quad a_3 = u_m, \quad a_{4,5} = u_m \pm \sqrt{gh}.$$
Example for $\beta$-regularization: $N = 3$

Matching coefficients yields

$$
\beta_1 = -\frac{7 \left(s^2 + gh\right)}{4s}, \quad \beta_2 = -\frac{7 \left(s^2 + gh\right)}{4s}, \quad \beta_3 = 0,
$$

$$
\beta_4 = \frac{21 \left(gh - s^2\right)}{20s} - \frac{2}{5}s \text{ and } \beta_5 = 0
$$

$$
A_{hyp} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-\frac{s^2}{3} - u_m^2 + gh & \frac{2s}{3} & 0 & 0 & 0 \\
-2su_m & 2s & u_m & \frac{3s}{5} & 0 \\
-\frac{1}{3} \left(2s^2\right) & 0 & \frac{s}{3} & u_m & \frac{4s}{7} \\
\frac{7(s^2+gh)u_m}{4s} & \frac{7(s^2+gh)}{4s} & 0 & \frac{21(gh-s^2)}{20s} & u_m \\
\end{pmatrix}
$$
Linearized around equilibrium

- SWME: stable for small non-equilibrium
- LSWME: stable for small non-equilibrium
- $\beta$-LSWME: unstable in equilibrium
Convergence of solutions for linear smooth test case

$h(t_{end}, x)$; IVP: $u_m_0 = 0.25; s_0 = -0.25; \kappa_0 = 0$

- Reference Solution
- $N=0$
- $N=1$
- $N=2$
- $N=3$
- $N=4$
- $N=5$
Figure: $L_1$-error $||\Delta h||$ and $||\Delta u_m||$ for LSWME using linear IVP and friction.
Convergence of solutions for quadratic smooth test case

![Graph showing convergence of solutions for quadratic smooth test case]

- Reference Solution
- N=0
- N=1
- N=2
- N=3
- N=4
- N=5

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\[ u(t, x, \zeta) = u_m(t, x) + \sum_{j=1}^{N} \alpha_j(t, x) \phi_j(\zeta) \]

Mean of \( u \) + Deviation of \( u \)

Next steps:

- Paper in preparation
- Alternative regularization ideas
- Non-conservative numerics investigation
- Application to mud flows
Summary of Shallow Water Moment Equations

\[ u(t, x, \zeta) = u_m(t, x) + \sum_{j=1}^{N} \alpha_j(t, x) \phi_j(\zeta) \]

Mean of \( u \)

Deviation of \( u \)

Next steps:
- Paper in preparation
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Thank you for your attention!
Convergence of solutions for linear smooth test case

<table>
<thead>
<tr>
<th>Friction coefficient</th>
<th>$\nu \in {0, 0.1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>slip length</td>
<td>$\lambda = 0.1$</td>
</tr>
<tr>
<td>initial mean velocity</td>
<td>$u_m(0, x) = 0.25$</td>
</tr>
</tbody>
</table>

**Domain**

| Temporal domain | $t \in [0, t_{end}]$, with $t_{end} \in \{0.2, 2.0\}$ |
| Spatial domain  | $x \in [-1, 1]$                                      |
| Spatial resolution | $n_x = 100$ and for the reference solution $n_x = 40$ |
| CFL number      | 0.5                                                  |

**Initial velocity profile (IVP) $u(0, x, \zeta)$**

| constant         | $u_m(0, x) = 0.25$                                  |
| linear           | $u_m(0, x) + s_0(x)\phi_1(\zeta) = 0.5\zeta$      |
| quadratic        | $u_m(0, x) + s_0(x)\phi_1(\zeta) + \kappa_0(x)\phi_2(\zeta)$ |
|                  | $= 0.15 + 0.6\zeta(1 - \zeta)$                    |

**Table:** Setup for simulation of SWME, LSWME, and reference solution.