Improving the Convergence of Moment Methods for Rarefied Gases Using Filters

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Motivation

Objective

Simulation of flow problems involving fast, rarefied gases

Knudsen number: \( Kn = \frac{\lambda}{L} \)
Moment Method \[ \text{Grad}, 1949 \]

**Boltzmann Transport Equation**

\[
\frac{\partial}{\partial t} f(t, x, c) + c \frac{\partial}{\partial x} f(t, x, c) = S(f)
\]

**Moment Method using Grad’s ansatz**

\[
f(t, x, c) = \frac{\rho}{\sqrt{2\pi\theta}} e^{-\frac{(c-v)^2}{2\theta}} \sum_{\alpha=0}^{M} f_{\alpha}(t, x) \phi_{\alpha}\left(\frac{c-v}{\sqrt{\theta}}\right)
\]

\[
\partial_t u_M + A(u_M) \partial_x u_M = S, \quad u_M = (\rho, v, \theta, f_3, f_4, \ldots, f_M)^T
\]
Hyberbolic Moment Models \([\text{Grad, 1949}]\)

Grad’s model is not hyperbolic \(\Rightarrow\) breakdown of solution

Use Hyperbolic Moment Model

1. HME \([\text{Cai et al., 2013}]\)
2. QBME \([\text{JK et al., 2013}]\)

\[
\partial_t \mathbf{u}_M + \tilde{\mathbf{A}}(\mathbf{u}_M) \partial_x \mathbf{u}_M = \mathbf{S}, \quad \mathbf{u}_M = (\rho, v, \theta, f_3, f_4, \ldots, f_M)^T
\]

- globally hyperbolic
- analytical eigenstructure
1D shock tube test [JK, 2017]

\[ \text{Kn} = 0.05, M = 4 \]

- accurate solution for small Kn
1D shock tube test [JK, 2017]

\[ Kn = 0.5, \, M = 4 \]

- accurate solution for small \( Kn \)
- increased applicability for larger \( Kn \)
1D shock tube test [JK, 2017]

\[ \text{Kn} = 0.5, \, M = 8, 9 \]

- accurate solution for small Kn
- increased applicability for larger Kn
- error reduction with increasing moments M
1D shock tube test [JK, 2017]

\[ Kn = 0.5, M = 4, \ldots, 9 \]

- accurate solution for small \( Kn \)
- increased applicability for larger \( Kn \)
- error reduction with increasing moments \( M \)
Starting point: Averaging

Kn = 1

Observation 1:
Convergence is improved by averaging odd and even moment solutions
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Questions
(1) Averaging times? $t_{\text{END}}$ or $t_k$
(2) Averaging weight? $u_{AV} = \alpha u_M + (1 - \alpha) u_{M+1}$
(3) Averaging multiple solutions? $u_{M+1}, u_M, u_{M-1}, \ldots$
(4) Averaging overhead reduction? $u_M$ and $u_{M+1}$
Linearized, collisionless moment model

\[ \partial_t \mathbf{u}_M + A \partial_x \mathbf{u}_M = 0, \]

Idea: Add artificial collision term in last equation

\[ \partial_t \tilde{\mathbf{u}}_M + A \partial_x \tilde{\mathbf{u}}_M = -\frac{1}{\epsilon} \tilde{\mathbf{Q}}, \quad \tilde{\mathbf{Q}} = (0, \ldots, 0, f_M)^T \]

1) \( \epsilon \to \infty \) \quad \Rightarrow \quad \tilde{\mathbf{u}}_M = \mathbf{u}_M

2) \( \epsilon \to 0 \) \quad \Rightarrow \quad f_M = 0 \Rightarrow \tilde{\mathbf{u}}_M = \mathbf{u}_{M-1}

3) \( 0 < \epsilon < \infty \) \quad \Rightarrow \quad \tilde{\mathbf{u}}_M \) in between \( \mathbf{u}_{M-1} \) and \( \mathbf{u}_M \)

Observation 2:
Artificial collision mimic averaging
Generalization of artificial collisions

Add artificial collision to every non-linear equation:

\[
\partial_t u_M + A(u_M) \partial_x u_M = -\frac{1}{\epsilon} \tilde{Q},
\]

Artificial collision terms:

\[
- \frac{1}{\epsilon} \tilde{Q}_i = - \frac{1}{\epsilon} \beta(i) f_i,
\]
Numerical solution

\[ \partial_t u_M + A(u_M) \partial_x u_M = -\frac{1}{\epsilon} \tilde{Q}, \]

Time splitting

1) \( \partial_t u_M + A(u_M) \partial_x u_M = 0, \)
2) \( \partial_t u_M = -\frac{1}{\epsilon} \tilde{Q} \)

Exact solution of collision step 2)

\[ \partial_t f_i = -\frac{1}{\epsilon} \beta(i) f_i, \]
\[ \Rightarrow f_i^{n+1} = \exp \left( -\frac{\Delta t}{\epsilon} \beta (i) \right) f_i^n \]
Filter function

\[
\Rightarrow f_{i}^{n+1} = \exp \left(-\frac{\Delta t}{\epsilon} \beta (i) \right) f_{i}^{n} \\
= \sigma (i, \Delta t) f_{i}^{n}
\]

Observation 3: Artificial collision can be solved by filtering
Filter function

\[ f_{i}^{n+1} = \exp \left( -\frac{\Delta t}{\epsilon} \beta (i) \right) f_{i}^{n} = \sigma (i, \Delta t) f_{i}^{n} \]

Observation 3:
Artificial collision can be solved by filtering

Time-consistent filter function:
[Di et al., 2017]

\[ \beta (i) = \begin{cases} 
0, & i \leq \frac{2M}{3}, \\
\left( \frac{i}{M} \right) \gamma, & i > \frac{2M}{3}.
\end{cases} \]

Filter strength: \( \epsilon \)

\[ M = 10, \, \epsilon = 1/36, \, \gamma = 36 \]
Filtered results, shock tube $Kn = 1$ (1)

(a) $M = 6$

(b) $M = 9$
Filtered results, shock tube $Kn = 1 (2)$

(c) $M = 12$

(d) $M = 15$
Filtered results, shock tube $Kn = 1$ (3)

(e) $M = 18$

(f) $M = 21$
Filtered convergence, shock tube

(g) $Kn = 1$

(h) $Kn = \infty$
## Conclusion

### Summary of FHME
- Averaging
- Artificial collision
- Filtering

### Benefits of FHME
- Improved convergence
- No computational overhead
- Easy implementation
Conclusion

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Thank you for your attention!
References

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