In [12]:
```python
# To add a new cell, type '# %%'
# To add a new markdown cell, type '# %% [markdown]'
# %%
from IPython import get_ipython
```

# DFT and FFT

## Discrete Fourier Transform (DFT)

Following [1,2], the *Discrete Fourier Transform* (DFT) transforms a sequence of $N$ complex numbers $\{\mathrm{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers, $\{\mathrm{X_k}\} := X_0, X_1, \ldots, X_{N-1}$ and is defined by:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

The *Inverse Discrete Fourier Transform* (IDFT) is defined by:

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot e^{\frac{i2\pi}{N}kn}$$

[1]: https://jakevdp.github.io/blog/2013/08/28/understanding-the-fft/ (https://jakevdp.github.io/blog/2013/08/28/understanding-the-fft/)

[2]: https://en.wikipedia.org/wiki/Discrete_Fourier_transform (https://en.wikipedia.org/wiki/Discrete_Fourier_transform)

## Comparison to Lecture

In the lecture, the vector of coefficients $\hat{y} := \left\{d_0, \ldots, d_{N-1}\right\}$ with

$$d_j = \frac{1}{N}\sum_{k=0}^{N-1} y_k e^{-jix_k} \qquad \text{for } j = 0 \ldots N-1$$

was defined as the DFT. However, we use the more common definition given above for this lab.

## Packages

- `IPython.display` : jupyter usage
- `numpy` : numeric calculations
- `matplotlib` : plotting
- `math` : for Python's mathematics functions
- `time` : for timings

In [13]:
```python
# Packages
from IPython.display import display
import numpy as np
import matplotlib.pyplot as plt
import math
import time
```

### Discrete Fourier Transform as Linear Operation

We can write the DFT

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

as linear operation (matrix-vector multiplication) using

$$\mathrm{X} = \mathrm{M} \cdot \mathrm{x}$$

where the matrix $\mathrm{M}$ is given by

$$(\mathrm{M})_{k,n} = e^{-\frac{i2\pi}{N}kn}$$

The IDFT

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{\frac{i2\pi}{N}kn}$$

can also be written as linear operation using

$$\mathrm{x} = \mathrm{m} \cdot \mathrm{X}$$

where the matrix $\mathrm{m}$ is given by

$$(\mathrm{m})_{n,k} = \frac{1}{N} e^{\frac{i2\pi}{N}kn}$$

Note that $\mathrm{m} = (\mathrm{M})^{-1} = \frac{1}{N}\overline{\mathrm{M}}^T$ (switched indices $k, n \mapsto n, k$ and complex conjugation $\overline{-i} \mapsto i$).

In [14]:
```
# Implement DFT and IDFT

def DFT(x):
    x = np.asarray(x, dtype=complex)
    N = x.shape[0]
    n = np.arange(N) # n = [0,...,N-1] (rows)
    k = n.reshape((N, 1)) # k = [[0],...,[N-1]] (cols)
    M = np.exp(-2j * np.pi * k * n / N)
    return np.dot(M, x)

def IDFT(X):
    X = np.asarray(X, dtype=complex)
    N = X.shape[0]
    k = np.arange(N) # n = [0,...,N-1] (rows)
    n = k.reshape((N, 1)) # k = [[0],...,[N-1]] (cols)
    m = 1/N * np.exp(2j * np.pi * k * n / N)
    return np.dot(m, X)

# Validation
# - Last weeks interpolation task [2,0,2,0] should give coeffs [1,0,1,0]
# - IDFT(IDFT(x)) == x
# - np.fft.fft(x) == DFT(x)

x = [2,0,2,0]
assert np.allclose(IDFT(x), [1,0,1,0])
x = np.random.rand(2**5)
assert np.allclose(IDFT(DFT(x)), x)
assert np.allclose(DFT(x), np.fft.fft(x))
assert np.allclose(IDFT(x), np.fft.ifft(x))
print("IDFT([2,0,2,0]) = [1,0,1,0] = ", IDFT([2,0,2,0]))
print("IDFT(DFT([2,0,2,0])) = [2,0,2,0] = ", IDFT(DFT([2,0,2,0])))
```

```
IDFT([2,0,2,0]) = [1,0,1,0] =  [1.+0.0000000e+00j 0.+6.1232340e-17j 1.-1.22464
68e-16j 0.+1.8369702e-16j]
IDFT(DFT([2,0,2,0])) = [2,0,2,0] =  [ 2.0000000e+00-1.22464680e-16j -1.2246468
e-16+2.99951957e-32j
  2.0000000e+00+1.22464680e-16j  1.2246468e-16+2.44929360e-16j]
```

In [15]:
```
np.round(IDFT(DFT([2,0,2,0])).real)
```

Out[15]:
```
array([ 2., -0.,  2.,  0.])
```

## Fast Fourier Transform (FFT)

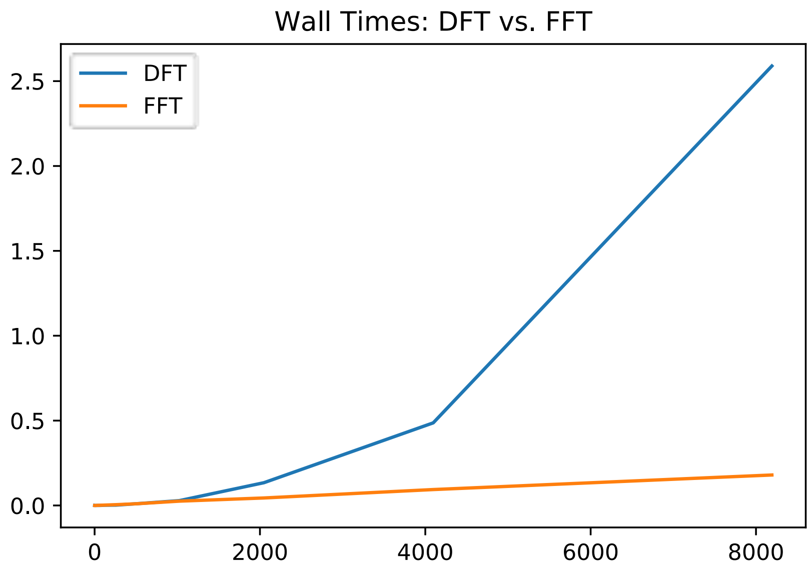
TODO: Explain derivation

In [16]:
```
# Implement FFT and IFFT
# Validate with DFT (question from lecture: there's no approx made!)

def FFT(x, cutoff=1):
    x = np.asarray(x, dtype=complex)
    N = x.shape[0]
    assert math.log2(N).is_integer()
    if N <= 1:
        return [x[0]]
    else:
        M = N//2
        x_even = [(x[k]+x[k+M]) for k in range(M)]
        x_odd = [(x[k]-x[k+M]) for k in range(M)]
        factor = [np.exp(-2*np.pi*1j*k/N) for k in range(M)]
        a = FFT(x_even) #! Note: Lecture has 1/2 here, cf. starting paragraph
        b = FFT(np.multiply(factor,x_odd))
        result = np.empty(len(a)+len(b), dtype=complex)
        result[::2] = a # even entries
        result[1::2] = b # odd entries
        return result # result = [a_0,b_0,...,a_M,b_M]

def IFFT(X):
  raise NotImplementedError

x = np.random.rand(2**5)
assert np.allclose(FFT(x), DFT(x))
print("np.allclose(FFT(x), DFT(x)) = ", np.allclose(FFT(x), DFT(x)))
```

```
np.allclose(FFT(x), DFT(x)) =  True
```

In [17]:
```
# Comparing the timings of DFT vs. FFT
N = 14
dft_times = np.empty(N)
fft_times = np.empty(N)
for i in range(N):
    # print("calculate i = ", i)
    x = np.random.rand(2**i)
    time1 = time.time()
    DFT(x)
    time2 = time.time()
    dft_times[i] = time2-time1
    time3 = time.time()
    FFT(x)
    time4 = time.time()
    fft_times[i] = time4-time3
dofs = [2**i for i in range(N)]
plt.plot(dofs, dft_times, dofs, fft_times)
plt.title("Wall Times: DFT vs. FFT")
plt.legend(["DFT", "FFT"])
```

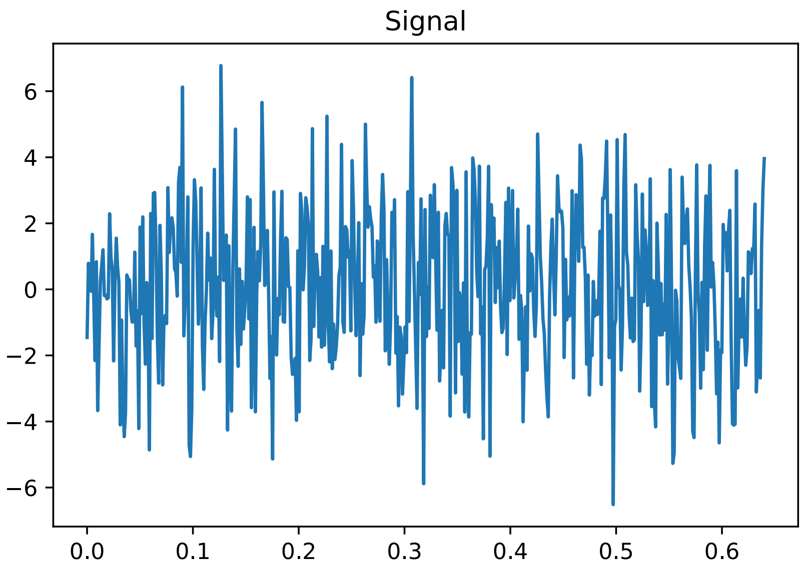
Out[17]:
```
<matplotlib.legend.Legend at 0x7fbdc8fc6208>
```

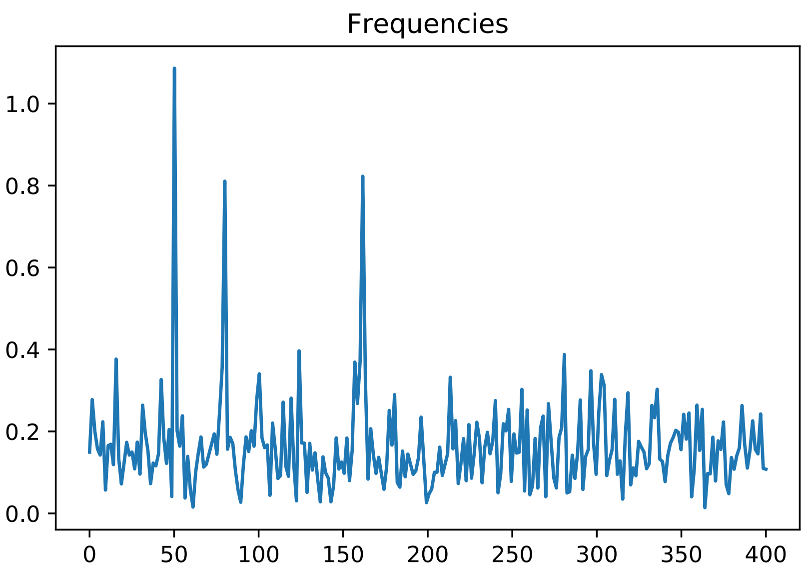



## Getting the Frequencies from a Discrete Signal

- TODO: Explain
- Source: [https://stackoverflow.com/questions/25735153/ (https://stackoverflow.com/questions/25735153/)](https://stackoverflow.com/questions/25735153/)

In [18]:
```
# Frecquency extraction
N = 2**9 # Number of samplepoints
T = 1.0 / 800.0 # sample spacing
x = np.linspace(0.0, N*T, N)
def signal(x):
    noise=np.random.normal(0, 2.0) # mean, var
    frequencies = [50, 80, 160]
    return noise + np.sum([np.sin(freq * 2.0*np.pi*x) for freq in frequencies])
data = [signal(x[i]) for i in range(len(x))]
yf = FFT(data)
xf = np.linspace(0.0, 1.0/(2.0*T), N//2)

plt.plot(x, data)
plt.title("Signal")
plt.show()

plt.plot(xf, 2.0/N * np.abs(yf[:N//2])) # stem-plot also nice
plt.title("Frequencies")
plt.show()
```



## Mutliplication vs. Convolution (Faltung) in Time vs. Frequency Space

Speedup in calculation.. TODO: Explain

In [19]:
```
# FFT-Multiplication
a = [4,3,2,1,0,0,0,0] # a=1234
b = [8,7,6,5,0,0,0,0] # b=5678
af = FFT(a)
bf = FFT(b)
cf = np.multiply(af, bf)
c = IDFT(cf)
powers = [10**(i) for i in range(len(a))]
cNumber = int(np.round(np.dot(c,powers).real)) # Round off erros occur => round
print(cNumber)
print("FFT-multiplication error = ", 1234*5678 - cNumber)
```

```
7006652
FFT-multiplication error =  0
```

## Further Optimizations Possible!

Link to other packages

In [20]:
```
# Speedtest
print("Compare our implementation with numpy:")
expo = 15
get_ipython().run_line_magic('time', 'FFT(np.random.rand(2**expo))')
get_ipython().run_line_magic('time', 'np.fft.fft(np.random.rand(2**expo))')
print("=> Numpy's np.fft.ftt is 1000x faster than ours 😒 ")
```

```
Compare our implementation with numpy:
CPU times: user 810 ms, sys: 10.2 ms, total: 820 ms
Wall time: 867 ms
CPU times: user 1.5 ms, sys: 100 µs, total: 1.6 ms
Wall time: 1.42 ms
=> Numpy's np.fft.ftt is 1000x faster than ours 😒
```

In [20]:
```

```