06-convection-diffusion

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1 Finite Differences for the Convection-Diffusion Equation

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The *Finite Difference Method* (FDM) is a numerical method to solve *Partial Differential Equations* (PDEs) approximately. We use second-order central finite differences for the diffusion term and investigate stability issue related to the discretization of the first-order advection term.

1.1 Convection-Diffusion Problem

We want to solve the Convection-Diffusion equation

$$\begin{aligned} -\epsilon\Delta u(x,y) + v\cdot\nabla u(x,y) &= f(x,y), & (x,y) &\in [0,L_x]\times[0,L_y] \\ u(x,0) &= b(x), & x &\in [0,L] \\ u(x,L) &= t(x), & x &\in [0,L] \\ u(L,y) &= r(y), & y &\in [0,L] \\ u(0,y) &= l(y), & y &\in [0,L] \end{aligned}$$

on a rectangle with $\epsilon > 0$ and $v \in \mathbb{R}^2$. The domain is discreitized structurally with N_x and N_y grid-points per dimension leading to $h_x = \frac{1}{N_x-1}$ and $h_y = \frac{1}{N_y-1}$.

[1]:
```julia
using LinearAlgebra
using SparseArrays
using PyPlot
using Plots
```

1.2 Domain and Boundary Conditions

The domain $\bar{\Omega}$ $[0,L_x]\times[0,L_y]$ is split into the nodes $\{(x_i,y_j)\}_{i,j=0\dots N} \in \bar{\Omega}_h$ with

$$x_i = \frac{L_x}{N_x-1}, \quad y_j = \frac{L_y}{N_y-1}$$

The interior nodes are $\{(x_i,y_j)\}_{i,j=1\dots N-1} \in \Omega_h$. We further need boundary conditions for all four sides of the unit square.

[2]:
```julia
struct Rectangle
    Lx::Float64
    Ly::Float64
    Nx::Int64
    Ny::Int64
    hx::Float64
    hy::Float64
    xh::Array{Float64,1}
    yh::Array{Float64,1}
    function Rectangle(Lx, Ly, Nx, Ny)
        hx = Lx/(Nx-1)
        hy = Ly/(Ny-1)
        xh = range(0, Lx, step=hx)
        yh = range(0, Ly, step=hy)
        N = new(Lx, Ly, Nx, Ny, hx, hy, xh, yh)
    end
end

struct UnitSquareUniform
    function UnitSquareUniform(N)
        h = 1/(N-1)
        xh = range(0, 1, step=h)
        yh = range(0, 1, step=h)
        N = Rectangle(1., 1., N, N)
    end
end

test = UnitSquareUniform(10)
display(test)

struct RectangleBCs
    bot
    right
    top
    left
end
```

```
Rectangle(1.0, 1.0, 10, 10, 0.1111111111111111, 0.1111111111111111, [0.0, 0.1111111111111111, 
```

1.3 Discretization Laplacian and First-Order Advection Term

We discretize the Laplacian using central finite differences with second order as

$$\Delta u(x,y) = \frac{-u(x+h,y) - u(x-h,y) + 4u(x,y) - u(x,y+h) - u(x,y-h)}{h^2} + \mathcal{O}(h^2)$$

such that the stencil for node values reads

$$u_{i,j} \approx \frac{-u_{i+1,j} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$
$$\Leftrightarrow [-\Delta u_h]_\xi = \frac{1}{h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \xi \in \Omega_h$$

We either use central finite differences or the upwind finite differences for the convection term (with $v_1^+ := \max(0, v_1)$ and $v_1^- := \min(0, v_1)$):

$$[v \cdot \nabla u_h^{up}]_\xi = \begin{bmatrix} 0 & \frac{v_2^-}{h_y} & 0 \\ \frac{-v_1^+}{h_x} & \frac{|v_1|}{h_x} + \frac{|v_2|}{h_y} & \frac{v_1^-}{h_x} \\ 0 & \frac{-v_2^+}{h_y} & 0 \end{bmatrix}$$

$$[v \cdot \nabla u_h^{central}]_\xi = \frac{1}{h^2}\begin{bmatrix} 0 & v_2 & 0 \\ -v_1 & 0 & v_1 \\ 0 & -v_2 & 0 \end{bmatrix} \text{(FIXME: only for } h_x = h_y\text{)}$$

TODO: Explain RHS vector and source vector

[3]:
```
function Δ (Ω::Rectangle)
       = kron
    dxx = spdiagm(-1=>ones(Ω.Nx-3), 0=>-2ones(Ω.Nx-2), 1=>ones(Ω.Nx-3))
    dyy = spdiagm(-1=>ones(Ω.Ny-3), 0=>-2ones(Ω.Ny-2), 1=>ones(Ω.Ny-3))
    return 1/Ω.hx^2 * I(Ω.Ny-2)   dxx + 1/Ω.hy^2 * dyy   I(Ω.Nx-2)
end

function v∘ (Ω::Rectangle, v, use_upwind)
       = kron
    if use_upwind
        v1dxLoc = 1/Ω.hx * spdiagm(
            -1=>-max(0,v[1])*ones(Ω.Nx-3),
            0=>abs(v[1])*ones(Ω.Nx-2),
            1=>min(0,v[1])*ones(Ω.Nx-3)
        )
        v2dyLoc = 1/Ω.hy * spdiagm(
            -1=>-max(0,v[2])*ones(Ω.Ny-3),
            0=>abs(v[2])*ones(Ω.Ny-2),
            1=>min(0,v[2])*ones(Ω.Ny-3)
        )
    else
        # central as default
        v1dxLoc = 1/Ω.hx^2 * spdiagm(
            -1=>-v[1]*ones(Ω.Nx-3),
            1=>v[1]*ones(Ω.Nx-3)
        )
        v2dyLoc = 1/Ω.hy^2 * spdiagm(
            -1=>-v[2]*ones(Ω.Ny-3),
            1=>v[2]*ones(Ω.Ny-3)
        )
    end
    return I(Ω.Ny-2)   v1dxLoc + v2dyLoc   I(Ω.Nx-2)
end

function b (Ω::Rectangle, f, bcs::RectangleBCs)
    Nx = Ω.Nx
    Ny = Ω.Ny
    xInt = Ω.xh[2:end-1]
    yInt = Ω.yh[2:end-1]

    fh = vec(f.(xInt,yInt'))
    bh = 1/Ω.hy^2 .* vec(bcs.bot.(xInt))
    rh = 1/Ω.hx^2 .* vec(bcs.right.(yInt))
    th = 1/Ω.hy^2 .* vec(bcs.top.(xInt))
    lh = 1/Ω.hx^2 .* vec(bcs.left.(yInt))

    bvec = zeros((Nx-2)*(Ny-2))
    bvec += fh
    bvec[1                  : 1      : Nx-2] += bh
    bvec[(Nx-2)*(Ny-2-1)+1  : 1      : end] += th
    bvec[(Nx-2)             : (Nx-2) : end] += rh
    bvec[1                  : (Nx-2) : end] += lh

    return bvec
end

function solveConvDiff(Ω::Rectangle, f, v,  , use_upwind, bcs::RectangleBCs)
    A = - *Δ (Ω) + v∘ (Ω, v, use_upwind)
    b = b (Ω, f, bcs)
    return (A) \ b
end

function plotSol(Ω::Rectangle, u, bcs::RectangleBCs, edgeAvg=true)
    pyplot()
    Nx = Ω.Nx
    Ny = Ω.Ny
    uMat = zeros(Nx,Ny)
    uMat[2:end-1,2:end-1] = reshape(u, (Nx-2, Ny-2))
    uMat[2:Nx-1,1] = vec(bcs.bot.(Ω.xh[2:end-1]))
    uMat[2:Nx-1,Ny] = vec(bcs.top.(Ω.xh[2:end-1]))
    uMat[Nx,2:Ny-1] = vec(bcs.right.(Ω.yh[2:end-1]))
    uMat[1,2:Ny-1] = vec(bcs.left.(Ω.yh[2:end-1]))
    if edgeAvg
        uMat[1,1] = 0.5 * (uMat[1,2] + uMat[2,1])
        uMat[1,Ny] = 0.5 * (uMat[1,Ny-1] + uMat[2,Ny])
        uMat[Nx,1] = 0.5 * (uMat[Nx-1,1] + uMat[Nx,2])
        uMat[Nx,Ny] = 0.5 * (uMat[Nx-1,Ny] + uMat[Nx-1,Ny])
    else
        uMat[1,1] = uMat[1,Ny] = uMat[Nx,1] = uMat[Nx,Ny] = 0
    end
    Plots.surface(Ω.xh, Ω.yh, uMat', camera=(35, 35), title="Plot")
end
```

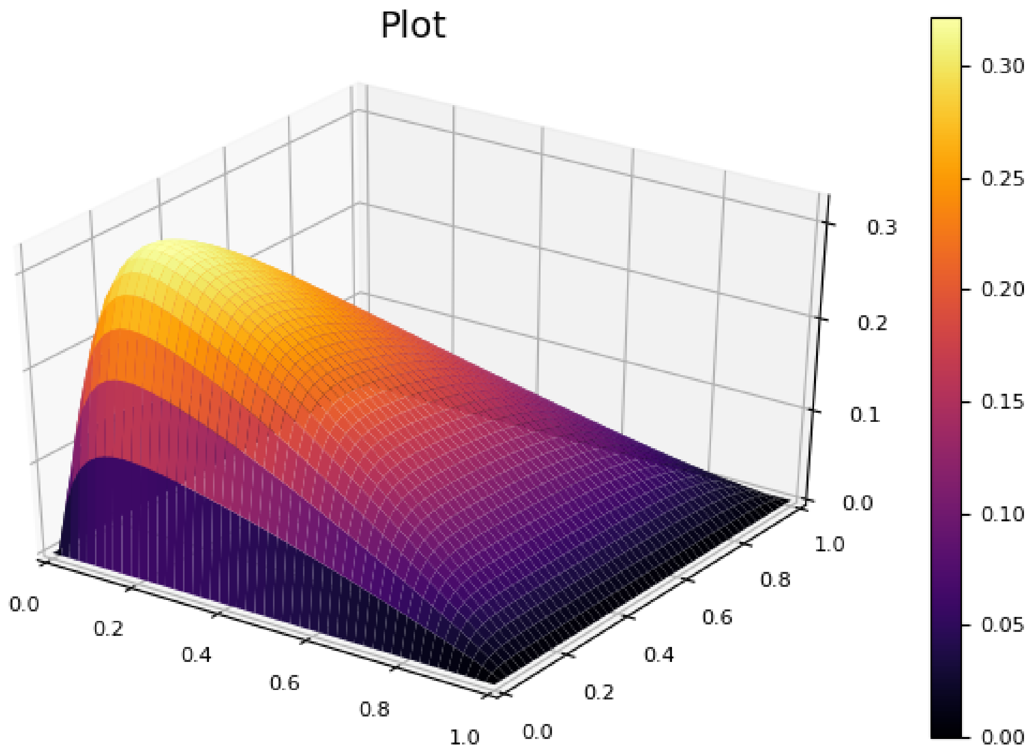
[3]:
```
plotSol (generic function with 2 methods)
```

1.4 Experiments

- Show failure of central diffs for convection-dominated cases
- Show success for upwind diffs for arbitrary cases

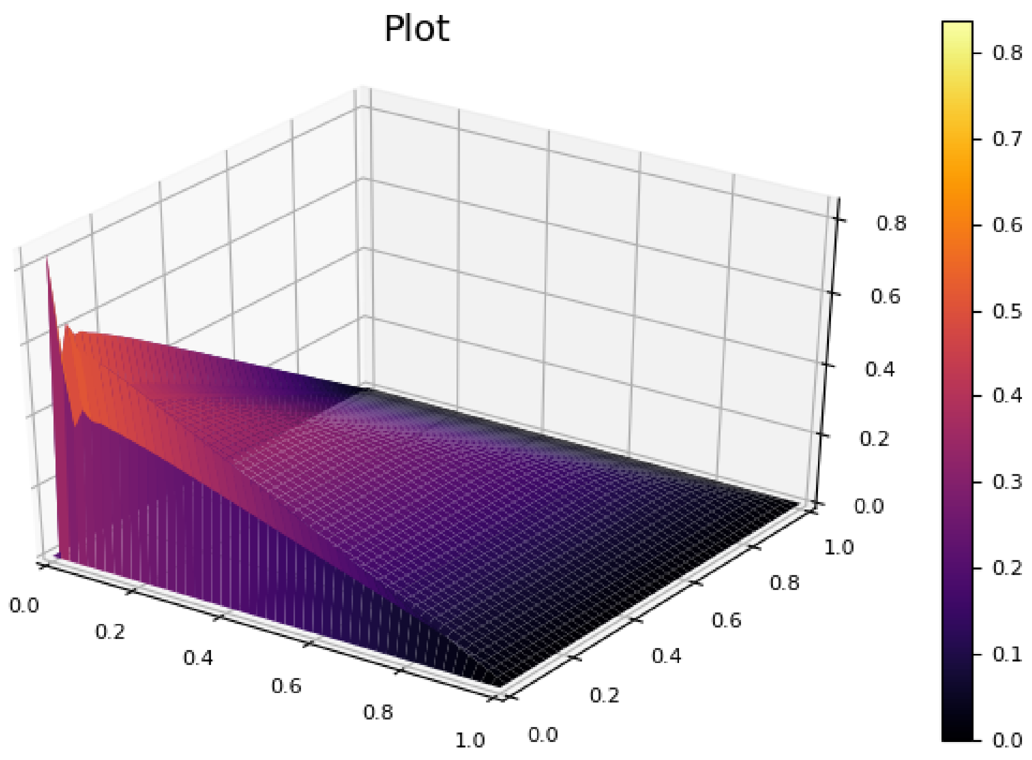
[4]:
```
Ω = UnitSquareUniform(50)
bcs = RectangleBCs(x -> 0, y -> 0, x -> 0, y -> 0)
f(x,y) = 10
v = [-0.2, -0.2]
use_upwind = false

u_stable = solveConvDiff(Ω, f, v, 1E0, use_upwind, bcs)
plotSol(Ω, u_stable, bcs)
```

[4]:



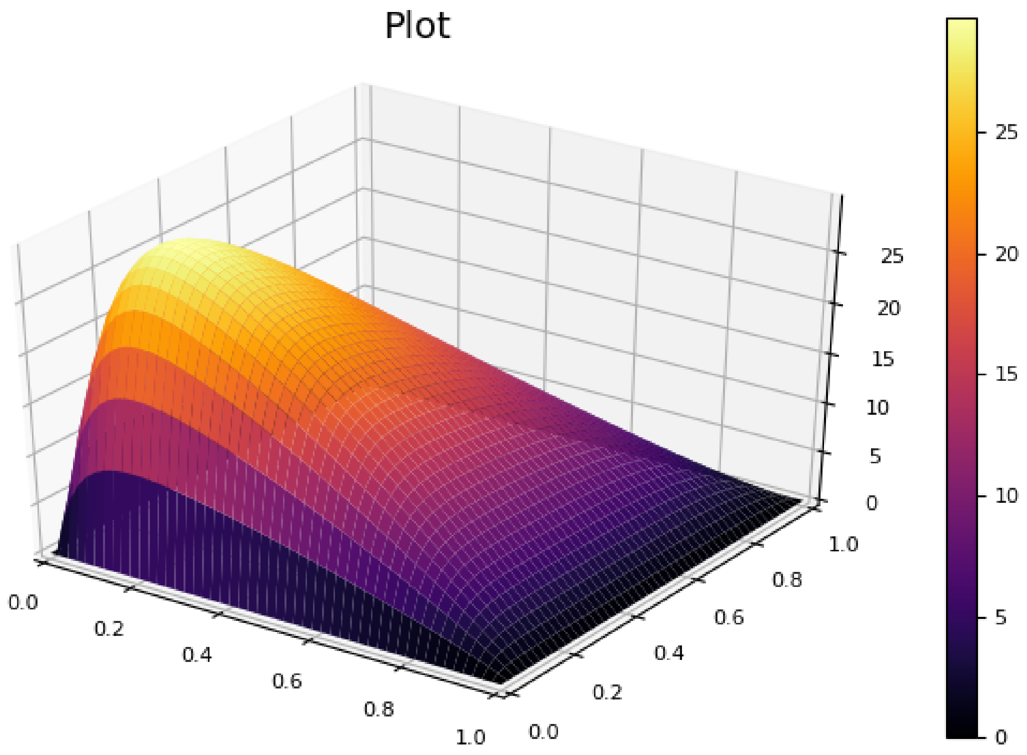
[5]:
```
u_instable = solveConvDiff(Ω, f, v, 1E-1, use_upwind, bcs) # lower epsilon/v␣
↪ratio
plotSol(Ω, u_instable, bcs)
```

[5]:



[6]:
```
Ω = UnitSquareUniform(50)
bcs = RectangleBCs(x -> 0, y -> 0, x -> 0, y -> 0)
f(x,y) = 10
v = [-0.2, -0.2]
use_upwind = true

u_stable = solveConvDiff(Ω, f, v, 1E-2, use_upwind, bcs)
plotSol(Ω, u_stable, bcs)
```

[6]:


Plot
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
0
5
10
15
20
25