Solenoidal Discrete Initialization for Locally Divergence-free MHD Simulations

R. Jeltsch* and M. Torrilhon**

ETH Zurich
Seminar for Applied Mathematics
CH-8092 Zurich, Switzerland


Summary. We propose a procedure to initialize the magnetic flux, possibly discontinuous, on a finite volume grid in an exactly solenoidal way. That is, a certain discrete divergence operator will vanish on each cell. Combined with a new locally divergence preserving numerical scheme we are able to conduct MHD simulations which have an exactly vanishing discrete divergence. In this paper we describe the new scheme and the initialization procedure and present the results of a simulation of a shock interaction with a magnetized cloud.

1 Introduction

The field of the magnetic flux $\mathbf{B}$ in magnetohydrodynamic simulations is subject to the constraint $\text{div} \, \mathbf{B} = 0$. To be precise, the divergence of $\mathbf{B}$ has to be zero initially and, there after, it is the character of the evolution equation for the magnetic flux which analytically preserves the divergence during the evolution. In MHD simulations this is no longer the case due to the discretization. One major task in the design of numerical methods for magnetohydrodynamics is the control of the divergence errors. Certainly, in the limit of finer grids any consistent scheme would reproduce the preservation properties of the exact evolution since it is an analytical property. However, MHD calculations are threatened by severe problems due to possible discontinuities which may generate divergence errors, see e.g. [2]. These errors usually accumulate and lead to a breaking down of classical numerical schemes making it impossible to calculate MHD solutions with those methods.

Recently, in [13] and [14] the construction of locally divergence preserving finite volume methods was presented. The first part of this paper will summarize the construction of these schemes for magnetohydrodynamics on

* mail: jeltsch@math.ethz.ch
** mail: manuel@math.ethz.ch
rectangular grids. The divergence preservation will be incorporated directly into the fluxes of the scheme reproducing the analytical preservation properties of the evolution. As a result the values of the discrete divergence operator given in Eqn. (16) will stay exactly the same during the entire calculation. Since the scheme is based only on special distributions of intercell fluxes, any finite volume scheme can be modified into a locally divergence preserving scheme. For the construction of the schemes we rely on the flux distribution framework given in [13] to formulate constraint preserving numerical methods. Flux distributions are piecewise constant basis shape functions in the grid for which a given discrete constraint vanishes. The update in a numerical scheme has to be built out of linear combinations of such flux distributions in order to provide constraint preservation. We demonstrate the flux distributions for divergence preservation and give the necessary modifications for finite volume fluxes.

The locally divergence preserving methods exactly preserve the discrete value of the divergence during the entire calculation. This implies special demands on the discrete initial conditions since its discrete divergence must vanish. Especially in the case of discontinuous data this is in general not easily obtained. In the second part of this paper we propose a new initialization procedure which provides directly solenoidal discrete fields. Using this initialization and the locally divergence-free scheme we present the simulation of a shock interaction with a magnetized cloud. The solution at different times is discussed and provides insight into the special behavior of MHD flows.

The paper is organized as follows: After we briefly introduce the equations of magnetohydrodynamics in Sec. 2, we discuss the major existing approaches of MHD divergence cleaning in Sec. 3. In Sec. 4 the deduction of locally divergence preserving schemes is summarized. Afterwards we discuss the problem of solenoidal initialization in Sec. 5 and present the simulation of the shock-cloud-interaction in the last section, Sec. 6.

2 Magnetohydrodynamic Equations

The equations of ideal magnetohydrodynamics consider the conservative variables density $\rho$, momentum density $\rho \mathbf{v}$, energy density $E$ and magnetic flux $\mathbf{B}$ to describe the flow of a plasma. As system of field equations we have

$$
\begin{align*}
\partial_t \rho + \text{div} \rho \mathbf{v} &= 0 \\
\partial_t \rho \mathbf{v} + \text{div} (\rho \mathbf{v} \mathbf{v} + (p + \frac{1}{2} \mathbf{B}^2) \mathbf{I} - \mathbf{B} \mathbf{B}) &= 0 \\
\partial_t E + \text{div} ((E + p + \frac{1}{2} \mathbf{B}^2) \mathbf{v} - \mathbf{B} \mathbf{B} \cdot \mathbf{v}) &= 0 \\
\partial_t \mathbf{B} + \text{div} (\mathbf{B} \mathbf{v} - \mathbf{v} \mathbf{B}) &= 0
\end{align*}
$$

(1)

that is, the balance laws of mass, momentum and energy and the induction equation. The system is closed by the equation of state of an ideal plasma

$$
E = \frac{1}{\gamma - 1} p + \frac{1}{2} \rho \mathbf{v}^2 + \frac{1}{2} \mathbf{B}^2
$$

(2)
where $\gamma$ is the adiabatic coefficient of the plasma. The system (1) forms a hyperbolic system of conservation laws (see e.g. [8] for hyperbolic properties) a fact which suggests to use a finite volume scheme in instationary MHD flow simulations.

The difficulty of such simulations is to handle the intrinsic constraint which follows by rewriting the induction equation (1) with a curl so that we have

$$\partial_t B + \text{curl}(B \times v) = 0 \quad \Rightarrow \quad \text{div } B = \text{const} \text{ in time (3)}$$

This means that the divergence of the magnetic flux remains untouched during the evolution. Since the magnetic flux has to be solenoidal in the initial conditions, it follows that it will be divergence-free for all times.

### 3 Divergence Cleaning Approaches

The construction of divergence-free methods for magnetohydrodynamics is vastly discussed in the literature. We may distinguish between three major approaches. See also the paper of Toth [15] for an overview and comparison of the major approaches.

The first one, originally described by Brackhill and Barnes in [2], uses a classical numerical method but cleans the field of the magnetic flux after every time step or after a certain number of time steps. The cleaning procedure solves the elliptic equation

$$\text{div grad } \psi = \text{div } \tilde{B} \quad \text{in } \Omega$$
$$\psi = 0 \quad \text{on } \partial \Omega$$

for the auxiliary discrete field $\psi$. The spoiled discrete field $\tilde{B}$ is corrected by $B_{i,j} = \tilde{B}_{i,j} + \text{grad } \psi|_{i,j}$. This method leads to solenoidal fields during the calculation and avoids divergence errors. However, the method is expensive due to the solution of a global elliptic equation. Moreover, inspection of the analytical equation shows that the preservation of the divergence is not connected to an elliptic problem. Analytically the divergence is locally preserved and it should be possible to construct a locally divergence preserving numerical method as well.

The second approach constructs divergence free methods by special discretization of the evolution equation of $B$. Originally described by Evans and Hawley in [6], these ideas have been used and further developed by Balsara and Spicer [1], as well as by Dai and Woodward [3]. In those methods again a correction step follows each time step of a classical numerical method. This correction considers the magnetic field components $b^{(x)}_{i,j + \frac{1}{2}}$ and $b^{(y)}_{i,j + \frac{1}{2}}$ stored at edges (in two dimensions), the so-called staggered grid. The relevant discrete divergence operator is given by
formulated with the staggered variables rather than the cell mean values. Each time step provides a divergence preserving evolution for the magnetic flux on the staggered grid. The operator (5) is exactly preserved. The correction character and the staggered grid appear as disadvantages since they spoil the cell average approach in the finite volume method. In his article [15] Toth showed that the staggered grid may be eliminated by explicit extrapolation and interpolation. The apparent restriction of the staggered approach to structured meshes is somewhat relaxed by DeSterck in [5].

The third approach is due to Powell [11] who constructed a modified analytical MHD system based on the assumption that $\text{div} \mathbf{B} \neq 0$. This new system contains additional terms which advect the divergence errors out of the computational domain. In [4] Dedner et al. and in [10] Munz et al. elaborate Powells ideas for MHD as well as for electrodynamics on unstructured grids. They introduce a new variable $\psi$ which is coupled to the system of MHD by

$$\frac{\partial}{\partial t} \mathbf{B} + \text{curl}(\mathbf{B} \times \mathbf{v}) + \nabla \psi = 0$$

$$\mathcal{D}(\psi) + \nabla \cdot \mathbf{B} = 0$$

where $\mathcal{D}$ is a linear operator. Different choices of $\mathcal{D}$ lead to either the advection method of Powell, an elliptic cleaning as in the first approach or a diffusive cleaning. However, Toth demonstrated in [15] that the artificial source terms arising in Powells system lead to wrong shock speeds in certain MHD Riemann problems.

All of the approaches are mainly concerned with eliminating an arising divergence error during the time steps such that the magnetic flux becomes solenoidal. However the solenoidal magnetic flux should not be seen as main issue. Inspired by the analytical properties, it is the update or residual in the numerical method that has to be divergence-free. Thus, the main problem is to construct numerical methods that exactly preserve the divergence during the evolution irrespective of the actual divergence of the field that is evolved. This, of course, will only be possible on a discrete level for a certain discretization of the divergence. For such an divergence preserving method, the solenoidality of the magnetic flux is a problem only for the initial data.

In [13] Torrilhon and Fey and in [14] Torrilhon worked out locally divergence preserving numerical schemes. These schemes use only one single finite volume grid to represent the fields, the updates and the divergence. In the following section we will shortly summarize the derivation of locally divergence preserving methods for MHD.

We restrict ourselves to the two-dimensional case. The extension of the presented algorithms to three dimensions is possible and mostly straightforward, see [14]. In two dimensions the divergence of the magnetic flux is only influenced by the components $B^{(x)}$ and $B^{(y)}$. Hence, the evolution of $B^{(z)}$ is not of interest for the divergence preservation.
4 Local Divergence Preservation

In [13] a general framework is given of numerical methods for general evolution equations with inherent constraints. In that framework a vector field \( \mathbf{u} \in \Omega \subseteq \mathbb{R}^D \) (\( D \): space-dimension) and a generic evolution

\[
\partial_t \mathbf{u} + \mathbf{F}(\mathbf{u}) = 0 \quad \text{in } \Omega
\]

with transport operator \( \mathbf{F} \) is considered. The generic constraint \( \mathcal{C} \) is assumed to be inherent to (7), that is the relation

\[
\mathcal{C}(\mathbf{F}(\mathbf{u})) \equiv 0
\]

holds, which directly implies

\[
\mathcal{C}(\mathbf{u}) = \text{const} \quad \text{in time}
\]

for any solution of (7). In the framework it is also assumed that the constraint is linear, which is fortunately the case in most applications, e.g. in MHD.

Here, we proceed with constructing divergence preserving schemes for magnetohydrodynamics on rectangular grids. Hence, the vector field is given by \( \mathbf{u} = \mathbf{B} \), and the constraint is \( \mathcal{C}(\mathbf{u}) = \text{div} \mathbf{B} \). The evolution equation is given by the induction equation (1). The computational domain \( \Omega \) is covered by a grid \( T \) with cells are denoted by \( K = (i, j) \) at positions \((x_i, y_j)\) and size \( \Delta x \times \Delta y \). In cases of accuracy considerations we refer to \( h = \max(\Delta x, \Delta y) \). A time discretization by \( \Delta t \) leads to a cell-wise constant grid function \( \mathbf{B}^n : T \rightarrow \mathbb{R}^D \) which approximates \( \mathbf{B} \) after \( n \) time steps by cell mean values. Cell averages of \( \mathbf{B} \) at time level \( n \) are denoted by \( \mathbf{B}^n_{i,j} \). Locally divergence preserving methods can also be constructed on triangular grids, see [14].

4.1 Flux Distribution Schemes

The central quantity of constraint preserving schemes is the so called “flux distribution”. It is the structure of the flux distribution that determines whether a certain scheme is constraint preserving or not.

**Definition 1 (flux distribution).** Given the space of vector-valued grid functions denoted by \( V = \{ g : T \rightarrow \mathbb{R}^D \} \), we define a “flux distribution” \( \Phi_K : V \rightarrow V \) which is attached to a grid cell \((i, j)\) and maps the grid function \( \mathbf{B} \) into another grid function, that is

\[
\Phi_{i,j}(\mathbf{B}) : T \rightarrow \mathbb{R}^D.
\]

The evaluation \( \Phi_{i,j}(\mathbf{B}) \big|_{k,l} \) gives the change of \( \mathbf{B} \) at cell \((k, l)\) caused by cell \((i, j)\), that is the flux.
A flux distribution is assigned to each cell of the grid and depends on the solution $\mathbf{B}$ in a local manner. The definition is more general than that of usual intercell fluxes, since it admits fluxes to any neighbouring cell, especially across corners. This incorporates multidimensionality from the very beginning. Conservation of $\mathbf{B}$ may be expressed by the statement that the integral over $\Phi_{i,j}(\mathbf{B})$ vanishes.

A certain form of the flux distribution and its dependency on $\mathbf{B}$ is usually constructed from consistency with the evolution equation, that is the induction equation in the present case. Once the flux distribution is defined an explicit evolution scheme follows by simply collecting contributions of all flux distributions, viz.

$$
\mathbf{B}_{i,j}^{n+1} = \mathbf{B}_{i,j}^n + \sum_{\text{cells } (k,l) \text{ surrounding } (i,j)} \Phi_{k,l}(\mathbf{B}_i^n)_{i,j}. \tag{11}
$$

The value of $\mathbf{B}$ in a cell $(i,j)$ is updated by contributions of all neighbouring cells. The contributions are given by evaluations of flux distributions. Note, that virtually any finite volume scheme can be written in the form (11), and the flux distribution may then be identified.

Since the divergence is linear we expect a discretization which may be written as matrix operation

$$
\text{div } \mathbf{B}_{i,j} = \text{div}_{i,j} \cdot \mathbf{B} + O(h^m) \tag{12}
$$
on the grid function $\mathbf{B}$. Here, $\text{div}_{i,j}$ represents the stencil of the discrete divergence operation in the grid. If preservation of the divergence should be achieved for the scheme (11) the following lemma gives sufficient conditions, see [13].

**Lemma 1.** If the flux distribution of the scheme is built by linear combinations of shape functions $\Phi_{i,j}^{(g)}$ which satisfy the condition

$$
\text{div}_{k,l} \cdot \Phi_{i,j}^{(g)} = 0 \quad \forall \text{ cells } (i,j),(k,l) \tag{13}
$$
then the resulting scheme (11) is exactly locally divergence preserving.

As the system (13) is homogeneous we generally hope for a solution space from which we only consider an appropriate basis set of shape functions $\Phi_{i,j}^{(g)}$ with $g = 1, 2, \ldots$ which all are constraint preserving. The final flux distribution has to be assembled from these solutions via

$$
\Phi_{i,j}(\mathbf{u}) = \sum_g \varphi_{i,j}^{(g)}(\mathbf{B}) \Phi_{i,j}^{(g)} \tag{14}
$$
with unknown coefficients $\varphi_{i,j}^{(g)}$, which give the amplitudes of the flux distributions. Note, that the choice of $\varphi_{i,j}^{(g)}$ does not affect the preservation of the
constraint. The expression for $\Phi_{i,j}$ enters the scheme (11) and the remaining coefficients $\varphi^{(g)}_{i,j}$ have to follow from consistency.

It follows from the lemma given above that the divergence preserving property of a scheme depends on the choice of the discrete divergence operator $\text{div}_{k,l}$ used in the conditions (13). Since the divergence can be evaluated discretely in many ways, certain numerical schemes will be preserve one discretization of the divergence but not another one.

4.2 Flux Distributions for Divergence Operators

We proceed to present solutions of the conditions (13), that is divergence preserving flux distributions for specific discretizations of the divergence.

In [13] it was shown that the classical divergence operator on a rectangular grid

$$\text{div}^{(0)}_{i,j} \mathbf{B} := \frac{B_{i+1,j}^{(x)} - B_{i-1,j}^{(x)}}{2\Delta x} + \frac{B_{i,j+1}^{(y)} - B_{i,j-1}^{(y)}}{2\Delta y}. \tag{15}$$

admit only a single flux distribution which does not give rise to practical schemes. However, the so-called extended operator $\text{div}^{(e)}_{i,j}$ given by

$$\text{div}^{(e)}_{i,j} \mathbf{B} := \frac{B_{i+1,j}^{(x)} - B_{i-1,j}^{(x)}}{2\Delta x} + \frac{B_{i,j+1}^{(y)} - B_{i,j-1}^{(y)}}{2\Delta y} \tag{16}$$

turned out to be more fruitful. Here, curled brackets stand for

$$\{\psi_{i,j}\}_y = \frac{1}{4} (\psi_{i,j+1} + 2\psi_{i,j} + \psi_{i,j-1})$$
$$\{\psi_{i,j}\}_x = \frac{1}{4} (\psi_{i+1,j} + 2\psi_{i,j} + \psi_{i-1,j}) \tag{17}$$

i.e. averaging between vertical or horizontal cells. Like the classical operator, $\text{div}^{(e)}_{i,j} \mathbf{B}$ gives a second order approximation to the divergence on cell $(i,j)$ using a $3\times3$ stencil. The difference lies only in the second order residual terms. It follows that both operators are equivalent up to second order for smooth solutions. Hence, if one operator is exactly preserved by a numerical scheme the other one will give a result of $O(h^2)$ in smooth regions of the flow. The exact control of a single operator is enough also in the case of non-smooth solutions in order to keep the solution free from divergence error.

At discontinuities the equivalence of different discrete operators is no longer true. Indeed, the difference between discrete operators is of order $O(1)$ due to the blow-up of the residual terms in the vicinity of discontinuities. However, if the scheme is divergence preserving for a specific divergence operator the value of the divergence given by this operator will be exactly zero also across discontinuities. Furthermore, any $O(1)$-result of the other operators will stick to the discontinuities, since behind and in front of discontinuities the solution is smooth and the operators give equivalent results. Thus, no divergence error will spoil the solution. To some extend, the $O(1)$-result of
non-preserved operators in divergence preserving schemes in the vicinity of shocks simply indicate the presence of blowing-up derivatives of $B$ and not the presence of a divergence error. Note, that for non-smooth solutions the operators (16) and (15) still give a result consistent to the divergence in the sense of distributions.

The possible flux distributions of $\text{div}^\ast$ follow from the solution of the homogeneous system given in (13). The system arises by fixing the cell $(i, j)$ and successive evaluation of the divergence operator in the neighbourhood of $(i, j)$ (see [13] for details). We obtain four possible shape functions for flux distributions, whose support consists of four cells. The non-vanishing values of the first flux distribution are given by

$$\hat{\Phi}_K^{(1)} = \begin{pmatrix} \ldots & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

and the others follow by translation as indicated in Fig. 1. From the flux distributions follow that the fluxes from one cell into another are not any more independent if we want to control the divergence. The sketch in Fig. 1 demonstrates how the fluxes are coupled. A flux from $K$ into its right neighbour, i.e. a change of the magnetic flux in that cell, immediately implies a flux into e.g. the upper right corner. If this coupling is not respected, the divergence will be spoiled.

The update in a numerical method which is constraint preserving must be built out of linear combinations of flux distributions $\hat{\Phi}_K^{(g)}$ as given in (18). Once such a scheme is constructed the local value of the discrete divergence operator $\text{div}^\ast$ in (16) will remain completely unchanged during the time steps. The values of the operator given by the discrete initial conditions will be exactly preserved.

### 4.3 Modification of Flux Distributions

Equipped with the information of the last section we will now modify a generic finite volume scheme (see e.g. [7]) for MHD such that it is locally divergence
Solenoidal Discrete Initialization for MHD

preserving. It is sufficient to consider only the part of the scheme updating the magnetic flux given by

\[ B_{i,j}^{n+1} = B_{i,j}^n + \Delta t \left( \frac{F_{i-\frac{1}{2},j} - F_{i+\frac{1}{2},j}}{\Delta x} + \frac{G_{i,j-rac{1}{2}} - G_{i,j+\frac{1}{2}}}{\Delta y} \right) \]

where \( F \) and \( G \) are magnetic flux components of intercell fluxes which are obtained using the full set of conservative variables. These intercell fluxes are assumed to be given by any Riemann solver (e.g. HLLE, Roe, ...). Clearly, these fluxes depend on all MHD variables of the adjacent cells. However, this dependency is suppressed in this section. Since the type of Riemann solver remains unspecified, the following modifications may be applied to virtually any finite volume scheme.

As first step the flux distributions \( \Phi^{\text{class}} \) of the classical scheme given in (19) are identified. Due to the curl-structure of the induction equation (3) the flux \( F \) in x-direction changes only the y-component of the magnetic flux and vice versa for the flux \( G \). Furthermore the amplitude of both intercell fluxes is given by a single scalar function \( f \), since the flux in x-direction and the flux in y-direction is governed by the same function in the induction equation. Hence, we write

\[ F_{i+\frac{1}{2},j} = -f_{i+\frac{1}{2},j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad G_{i,j+\frac{1}{2}} = f_{i,j+\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

The flux distributions are most easily defined on the cell interfaces. The definitions

\[ \Phi^{\text{class}}_{i+\frac{1}{2},j} \bigg|_{i,j} = f_{i+\frac{1}{2},j} \begin{pmatrix} 0 \\ \Delta y \end{pmatrix} \quad \Phi^{\text{class}}_{i+\frac{1}{2},j} \bigg|_{i+1,j} = f_{i+\frac{1}{2},j} \begin{pmatrix} 0 \\ -\Delta y \end{pmatrix} \]  

\[ \Phi^{\text{class}}_{i,j+\frac{1}{2}} \bigg|_{i,j} = f_{i,j+\frac{1}{2}} \begin{pmatrix} -\Delta x \\ 0 \end{pmatrix} \quad \Phi^{\text{class}}_{i,j+\frac{1}{2}} \bigg|_{i,j+1} = f_{i,j+\frac{1}{2}} \begin{pmatrix} \Delta x \\ 0 \end{pmatrix} \]

lead to the equivalent flux distribution formulation

\[ B_{i,j}^{n+1} = B_{i,j}^n + \Delta t \left( \Phi^{\text{class}}_{i+\frac{1}{2},j} + \Phi^{\text{class}}_{i-\frac{1}{2},j} + \Phi^{\text{class}}_{i,j+\frac{1}{2}} + \Phi^{\text{class}}_{i,j-\frac{1}{2}} \right) \bigg|_{i,j} \]

of the scheme (19). So far nothing has happened except a reformulation of the finite volume scheme. A flux distribution of the classical scheme is depicted at the left hand side of Fig. 2. Note that the evaluation of the divergence on neighboring cells in the picture will vanish neither using \( \text{div}^{(0)} \) nor \( \text{div}^{(x)} \), hence the classical scheme (19) does not preserve the divergence, as expected.

A divergence preserving scheme may be established by modifying the flux distributions such that they form linear combinations of the shape functions \( \Phi^{(1)} \) given in the previous section. The difficulty is to obtain a consistent method. We suggest to use

\[ \Phi^{\text{div}}_{i,j+\frac{1}{2}} = -\frac{1}{2} f_{i,j+\frac{1}{2}} \left( \Phi^{(1)}_{i,j} + \Phi^{(2)}_{i,j} \right) \]
Fig. 2. Left: the flux distribution of a classical finite volume scheme applied to the induction equation. This flux distribution is not divergence preserving. Right: a modified flux distribution based on divergence preserving shape functions as shown in Fig. 1.

as divergence preserving flux distribution. Its sketch may be found at the right hand side of Fig. 2. The flux distribution $\Phi_{i,j+1/2}^{\text{div-free}}$ is built analogously. These flux distributions use the same amplitude of the intercell fluxes as the classical distribution except they distribute this flux on more cells. The enlarged support results in a more lengthy formulation of the scheme, since the value $B_{i,j}^{n+1}$ is influenced by intercell fluxes of neighbouring cells. The scheme with divergence preserving flux distributions read

$$B_{i,j}^{n+1} = B_{i,j}^n + \frac{\Delta t}{\Delta x \Delta y} \left( \Phi_{i,1/2,j+1/2}^{\text{div}} + \Phi_{i-1/2,j+1/2}^{\text{div}} + \Phi_{i+1/2,j-1/2}^{\text{div}} + \Phi_{i-1/2,j-1/2}^{\text{div}} \right)_{i,j}$$

and

$$B_{i,j}^{n+1} = B_{i,j}^n + \frac{\Delta t}{\Delta x \Delta y} \left( \Phi_{i+1/2,j+1/2}^{\text{div}} + \Phi_{i-1/2,j+1/2}^{\text{div}} + \Phi_{i+1/2,j-1/2}^{\text{div}} + \Phi_{i-1/2,j-1/2}^{\text{div}} \right)_{i,j}.$$  (25)

Obviously the new scheme has a larger stencil and is expected to introduce slightly more diffusion into the numerical solution. However a significant decrease of resolution has not been observed in the numerical experiments. It is also important to note that the preservation of the divergence requires the coupling of the changes in the neighbouring cells and leads necessarily to a larger stencil. For the same reason the preserving scheme appears with a multidimensional flavour. It becomes evident that multidimensionality is a key issue to control the divergence constraint.

To show consistency of the divergence preserving scheme we make the flux distributions in (25) explicit and rearrange the resulting terms. Finally we obtain the equivalent formulation

$$B_{i,j}^{n+1} = B_{i,j}^n + \Delta t \left( \frac{1}{\Delta x} \left( \langle f_{i,j-1/2} \rangle - \langle f_{i,j+1/2} \rangle \right) \right)$$

where the angular brackets stand for the averaging of certain neighbouring intercell fluxes, see [14] for details. Assuming that the intercell flux amplitudes $f_{i+1/2,j}$ are at least second order approximations to the exact values, e.g. by linear reconstruction, we proceed with a Taylor expansion of the above given scheme. Finally this leads to the statement.
\[ \left( \frac{1}{2h^2} (\langle f_{i,j} - \frac{1}{2} \rangle - \langle f_{i,j} + \frac{1}{2} \rangle) \right) = \left( \frac{-\partial_y f}{\partial_x f} \right)_{i,j} + O(h^2) \]  

which shows second order consistency in space with the induction equation (3) if \( f \) is substituted by \((v \times B)(z)\). Second order in time may now be obtained by Runge-Kutta integration of the residual.

5 Discrete Initial Conditions

The new locally divergence-free scheme preserves the value of the divergence in each time step exactly. Any field of divergence from the initial conditions will be frozen during the calculation. Hence, the requirement \( \text{div} \, B = 0 \) is a problem of the initial conditions: Like in the analytical case a (discrete) solenoidal field remains solenoidal. Usually, the analytical prescription of the initial values of the magnetic flux is divergence-free. However, the discretization of these initial conditions, especially discontinuous ones, may introduce divergence errors which spoil the entire simulation. We will discuss this problem in the case of a Riemann problem and propose a solution.

5.1 Example: Riemann Problem

For simplicity let us consider only the induction equation for \( B \). The velocity field shall be assumed to be constant and given by

\[ v = v_0 \begin{pmatrix} \sin \varphi \\ \cos \varphi \end{pmatrix} \]  

pointing in the direction of an angle \( \varphi \). Then, the induction equation becomes an equation for the \( B \)-field alone. The initial magnetic flux is prescribed in the form of a Riemann problem

\[ B(x, t = 0) = \begin{cases} B_0 & x < -y \tan \varphi \\ B_1 & x > -y \tan \varphi \end{cases} \]  

where the line of discontinuity is perpendicular to the velocity vector. The vectors \( B_0, 1 \) are constant. As a result the problem is essentially one-dimensional in the direction \((\cos \varphi, \sin \varphi)^T\). The solution of this Riemann problem is simply the advection of the discontinuity by the velocity field.

As example we consider the values \( B_0 = R(\varphi)(1, 1)^T \) and \( B_1 = R(\varphi)(1, 2)^T \) where the rotation matrix

\[ R(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \]  

rotates the vectors such that the normal component across the discontinuity is constant. Hence, the initial conditions have zero divergence in the weak sense.
Fig. 3. Sketch of the distribution of the cell values of $B^2$ and of $\text{div}^{(s)} B$ in the vicinity of the discontinuity given by the initial conditions (29) in the case of $\tan \varphi = \frac{1}{2}$. The values of $\text{div}^{(s)} B$ in the cells along the discontinuity tend to $\pm \infty$ if the grid is refined.

If we calculate the solution to this Riemann problem with the locally divergence-free scheme for the angle $\varphi = 0$ or $\varphi = \frac{\pi}{2}$ the simulation runs through smoothly and the result gives an approximation to the propagating discontinuity. The value of the divergence operator $\text{div}^{(s)}$ gives exactly zero for all time steps.

However, if we turn to intermediate values of $\varphi$, e.g. $\tan \varphi = \frac{1}{2}$, the situation is different. Already the initialization of the grid cells leads to a significant divergence error. At the left hand side of Fig. 3 the initial distribution of the cell values for $B^2$ is sketched for the case $\tan \varphi = \frac{1}{2}$. Each cell is constantly initialized by the value of $B$ of the initial conditions at the cell center. We remark, that an integration of the initial conditions on each cell will not essentially change the situation. The corresponding values of the divergence evaluated by the extended operator $\text{div}^{(s)}$ on each cell is shown on the right hand side of Fig. 3. Along the discontinuity the divergence gives a pattern of alternating large positive and negative values. Due to the divergence preservation property of our numerical scheme this divergence field will be exactly frozen and stay in place during the calculation. The result for such a calculation is shown at the left hand side of Fig. 4. The grid consists of 40x40 cells in the domain $[-1,1]^2$ and the result is shown after 10 time steps. Note, that the pattern of the non-zero divergence spoils clearly the field of $B$ and of $B^2$ during the simulation. A similar calculation based on the complete MHD system would immediately fail.
Fig. 4. Results for the Riemann problem given in (29) for the case $\tan \varphi = \frac{1}{2}$ after 10 time steps. Left: direct initialization of the magnetic flux leads to a spurious solution spoiled due to non-zero divergence. Right: the true solution is recovered by use of solenoidal discrete initialization.

5.2 Solenoidal Initialization

In order to get rid of the divergence in the initial conditions, the discrete initial field has to be corrected.

A first idea is to use the global cleaning approach as proposed e.g. in [2]. We stress that this cleaning procedure would only be needed for the initial conditions with non-vanishing divergence due to discontinuities. Hence, the procedure is only applied once in the beginning of the calculation. The cleaning procedure solves the elliptic equations (4) for the auxiliary discrete field $\psi$. The discrete initial field $\tilde{B}$ is afterwards corrected by $B_{i,j} = \tilde{B}_{i,j} - \nabla \psi_{i,j}$ which gives a solenoidal field.

The differential operator $\nabla \nabla$ has to be built from the extended divergence operator (16) since the result should give a divergence-free field according the extended operator. The use of the traditional discretization of the Laplace operator will not result in this property. The construction of the Laplace operator by applying an appropriate discrete gradient and afterwards $\nabla \nabla$ to the field $\psi$ results in a special discrete Laplace operator which assures that the evaluation of $\nabla \cdot (\psi \nabla \cdot)$ on the corrected solution will be zero. The discretized form of (4) may be solved by using iterative linear solvers.

This elliptic cleaning of the discrete initial conditions has been implemented and lead to satisfactory results, see [13]. However, it is quite expensive, though it is only applied once at the beginning of the calculation. Furthermore, we note that the analytic initial conditions need no cleaning. We would like to use the solenoidal character of the initial conditions in order to obtain a directly solenoidal initialization. The following lemma turns out to be useful in this situation. It relates the extended divergence operator to the operator
formulated with the staggered grid, that is magnetic field components $b_i^{(x)}$ and $b_j^{(y)}$ which are stored at edges.

Lemma 2. If $\text{div}(0) b_{i,j} = 0 \ \forall i,j$, we have the statement

$$B_{i,j} = \frac{1}{2} \left( b_i^{(x)} + b_i^{(y)} + b_i^{(y)} - b_i^{(y)} \right) \ \forall i,j \Rightarrow \text{div}(B) = 0 \quad (31)$$

for the extended divergence operator.

Hence, if we have staggered variables $b_i^{(x)}$ and $b_j^{(y)}$ in the grid which have a zero discrete divergence, the cell mean values $B_{i,j}$ obtained by averaging out of these staggered variables will have zero divergence measured with the extended operator. This is exactly what is needed for the initialization. It remains to construct a solenoidal distribution of staggered variables. We use the analytic representation

$$B(x,y) = \begin{pmatrix} \partial_y \phi(x,y) \\ -\partial_x \phi(x,y) \end{pmatrix} \quad (32)$$

with a potential $\phi(x,y)$, which is possible thanks to $\text{div} B = 0$. With this representation the staggered variables are given by

$$b_i^{(x)} = \frac{\partial_x \phi_i^{(x)} + \partial_x \phi_i^{(y)} - \partial_y \phi_i^{(y)} - \partial_y \phi_i^{(y)}}{\Delta y}, \quad b_j^{(y)} = \frac{\partial_y \phi_i^{(x)} + \partial_y \phi_i^{(y)} - \partial_x \phi_i^{(y)} - \partial_x \phi_i^{(x)}}{\Delta x} \quad (33)$$

which may be inserted into the formula for $B_{i,j}$ given in (31). Finally, the initial cell mean values are obtained depending on the values of $\phi$ in the four corners of the cell, which are easily evaluated, since $\phi(x,y)$ is a function explicitly known from the initial conditions.

For the piecewise constant Riemann data (29) we use

$$\phi(x,y) = B^{(x)}_i y - B^{(y)}_i x \quad (34)$$

in each domain of the initial condition. Since the normal component of $B$ is constant, $\phi(x,y)$ is continuous across the discontinuity of $B$. The evaluation of the extended divergence operator on the initial distribution gives now exactly zero in every cell. The calculation of 10 time steps with the divergence-free initial conditions is shown at the right hand side of Fig. 4. It is free from divergence errors.

6 Simulation of a Shock-Cloud-Interaction

To conclude this paper we present the simulation of a shock interaction with a plasma cloud. This simulation will demonstrate the abilities of the divergence-preserving scheme as well as provide some insight into the physical behavior
Fig. 5. Initial configuration for the simulation of a shock interaction with a magnetized dense plasma cloud. A normal shock with Mach number 8 is located at \( x = -0.6 \). The cloud has a radius of \( r_0 = 0.4 \) and is in equilibrium with the surrounding.

of plasma flows. The results of the last section are applied in order to obtain a directly divergence-free initialization.

Consider the configuration sketched in Fig. 5. At position \( x = -0.6 \) a shock is located which travels to the right with a Mach number \( M_s = 8 \) calculated for the right hand state. The domain right to the shock is at rest with pressure and density equal to unity. At both sides of the shock the magnetic flux vanishes, thus the state to the left of the shock is given by the Rankine-Hugoniot-conditions of the Euler equations. Inside the circle with radius \( r_0 \) around the origin we construct a magnetized cloud of dense plasma. This cloud is at rest as well, but its density is twice as large as in the surrounding. The magnetic field lines form circles. For the distribution of the magnetic flux \( \mathbf{B} \) we assume

\[
\mathbf{B}_{\text{inside}} = B_{\text{max}} \frac{r}{r_0} \mathbf{e}_\varphi \tag{35}
\]

where \( \mathbf{e}_\varphi = (-y/r, x/r)^T \) is the unit vector in \( \varphi \)-direction. The boundary of the cloud represents a tangential discontinuity in which the tangential component of \( \mathbf{B} \) drops to zero. The plasma cloud shall be in equilibrium with the surrounding, hence the complete plasma pressure \( p^\star \) must be constant. It follows the relation

\[
p_{\text{inside}} = p_{\text{outside}} = \frac{1}{2} B_{\text{inside}}^2 \tag{36}
\]

for the pressure inside the cloud. Note that there is a pressure decrease across the boundary of the cloud due to the discontinuous magnetic flux. Hence, the cloud may be interpreted as a bubble with surface tension. This will also become visible in the simulation.

Since the initial conditions of \( \mathbf{B} \) are discontinuous we use the initialization procedure presented in the last section. The potential of the magnetic flux is given by
Fig. 6. Shock interaction with a magnetized cloud at time $t = 0.3$. Left: density contours and contour lines. Right: pressure contours and contour lines. The field lines of the magnetic flux are superimposed in the pressure plot.

\[
\phi(x, y) = \begin{cases} 
B_{\text{max}} \frac{r_0^2 - (x^2 + y^2)}{2r_0} & x^2 + y^2 < r_0^2 \\
0 & \text{else}
\end{cases}
\] (37)

which represents the distribution (35). In the simulation presented we used $B_{\text{max}} = 1.3$ and $r_0 = 0.4$. The simulation was conducted up to the time $t = 0.5$. Due to symmetry only the upper half of Fig. 5 needs to be calculated. For the boundaries to the left and right and at the top constant extrapolation is used. To reduce the influence of spurious reflections the computational domain has been expanded in $y$-direction. The complete domain simulated was $[-0.8, 4.2] \times [0, 2]$. At $t = 0.5$ the shock is approximately at $x = 4.4$, that is it has shortly left the domain.

The computational domain was discretized with $1000 \times 2500$ cells. The maximal allowed Courant number was chosen to be 0.9 and the time step was adjusted adaptively. The intercell flux was computed by the HLLE Riemann solver as described in [12] or [16], while the linear reconstruction was limited using the WENO limiter of [9]. Second order time integration is done by Heun’s method. The implementation of the divergence-preserving scheme has been parallelized via MPI and the simulation was conducted on 6 PC’s with 2.4GHz CPU’s. The entire simulation used 4500 time steps and took approximately 35 hours.

In Fig. 6 the pressure and density fields are displayed at $t = 0.3$. Superimposed to the contours of the pressure at the right hand side the field lines of $B$ are drawn. The shock has just passed the cloud. The flow field around the cloud is dominated by the reflected shock created by the incident shock hitting the cloud and spreading out to all sides. The cloud is strongly compressed and accelerated. It will continue drifting in the flow behind the shock. Along the
incident shock six Mach reflection points are visible each consisting of three shocks and a slip line. The two most inner reflection points travel along the incident shock line towards each other while the other four reflection points travel outwards. Inside the cloud a complicated shock system has formed consisting of several magnetohydrodynamic shocks in which the magnetic field lines are bent. Note that the field lines essentially follow the deformation of the cloud. The magnetic field lines of $B$ stick to the matter.

However, for physical reasons the magnetic flux goes against the deformation and tries to restore a circular shape. This is similar to a bubble whose surface tension leads to a spherical shape. Hence, the stickiness of the field lines leads to a restoring force to the matter of the cloud. At the left hand side of Fig. 7 we see the contours of the density field at time $t = 0.5$ with the field lines of $B$ superimposed. At the right hand side the result for the case of initially vanishing magnetic flux inside the cloud is shown. It is obtained from a second simulation using the Euler equations alone. In the MHD case on the left hand side the cloud is re-contracting in order to restore a circle. It has lost some magnetized matter due to the strong vorticity and the upper and lower tips. This vorticity leads to a ongoing deformation in the case of the pure Euler equations. Also, the two strong kinks in the back of the cloud visible in Fig. 6 for the MHD case are still present. In the MHD case the magnetic forces let these kinks disappear.

During the entire calculation the divergence of $B$ evaluated with the extended divergence operator $\text{div}^\text{(x)}$ gives locally exactly the same value as in the initial conditions due to the divergence preservation property of our flux distribution scheme. Thanks to the initialization of $B$ which was done com-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{Simulation results for the shock interaction with a cloud at time $t = 0.5$. Left: magnetized cloud. Density contours with magnetic field lines superimposed. Right: non-magnetized cloud. Density contours. In both plots the contour levels have the range 3.5 - 6.5.}
\end{figure}
pletely divergence-free by use of (31) and (33) we have $\text{div}^{(*)} \mathbf{B} = 0$ for all times up to machine precision.

7 Conclusions

We presented new flux modifications that turn an arbitrary MHD finite volume method into a locally divergence preserving scheme. That scheme preserves the value of a certain discrete divergence operator and keeps the calculation free of divergence errors. We deduced the necessary modifications for rectangular in two dimensions using the flux distribution framework. Details and generalization of this approach to triangles may be found in [13] and [14]. The new scheme preserves exactly the value of the divergence as given by discrete initial conditions. We proposed an easy method how to initialize the magnetic flux in the grid such that the discrete divergence vanishes exactly. The abilities of the new scheme and initialization are demonstrated by the simulation of a shock interaction with a plasma cloud.

References


