Abstract. A general framework for constructing constraint-preserving numerical methods is presented and applied to a multidimensional divergence-constrained advection equation. This equation is part of a set of hyperbolic equations that evolve a vector field while locally preserving either its divergence or its curl. We discuss the properties of these equations and their relation to ordinary advection. Due to the constraint, such equations form model equations for general evolution equations with intrinsic constraints which appear frequently in physics.

The general framework allows the construction of numerical methods that preserve exactly the discretized constraint by special flux distribution. Assuming a rectangular, two-dimensional grid as a first approach, application of this framework leads to a locally constraint-preserving multidimensional upwind method. We prove consistency and stability of the new method and present several numerical experiments. Finally, extensions of the method to the three-dimensional case are described.

Key words. multidimensional hyperbolic equations, advection, constraints, finite-volume method, stability

AMS subject classifications. 65M06, 65M12

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1. Introduction. Many evolution equations in physics and engineering come with intrinsic constraints, i.e., local differential constraints that follow directly from the evolution operator. Such equations will be called constraint-preserving. The most popular example is the evolution of the magnetic flux density \( \mathbf{B} \) in electrodynamics: The divergence of \( \mathbf{B} \) must be zero for the initial conditions; afterwards the analytic evolution will keep the divergence of the field untouched. The same property is present in the system of magnetohydrodynamics of plasma flows (e.g., [6]), and a similar operator arises in the vorticity equation of incompressible flow (e.g., [12]). Vorticity-preserving equations are used, e.g., in meteorological flows [19], while in [20] a vorticity-preserving system is investigated, which may be related to the linearized Euler equations. Furthermore, the evolution equations of general relativity possess constraints whose properties are lively discussed; see, e.g., [22].

Intrinsic constraints are also expected to hold in numerical calculations of the corresponding evolution, at least in a discrete manner. The discrete approximation of the evolution operator should mimic the analytic properties as far as possible in order to obtain a most physical discrete solution. Nevertheless, the construction of commonly available numerical methods ignores constraints and indeed those methods generally introduce disturbances to the constraints. These disturbances may be argued to be small due to consistency: Since the constraint is an analytic property of the evolution operator it will be respected in a converged solution (see [3] in the context of general relativity). But this argument holds only for smooth solutions, where the disturbances of the constraint are of the order of the truncation error. For discontinuous solutions the error of the constraint becomes so large that computations completely fail (see, e.g., [6] in the context of magnetohydrodynamics). In
general, relativity errors in the constraints can excite instabilities [22]. It becomes obvious that controlling intrinsic constraints in numerical methods is required in the construction of accurate and reliable schemes. Even if the constraint is not preserved by the complete evolution operator but only by a part of it, a corresponding partial constraint-preserving discretization is most desirable. This yields that the constraint is numerically only affected by those causes which arose from the discretization of the nonpreserving part in the equations. A similar statement may also be found in [27].

The literature provides many works which deal with the divergence-constraint in the equations of magnetohydrodynamics. Global approaches like in [2] solve elliptic equations each time step in order to correct the solution. A popular approach uses a local correction procedure with the help of a staggered grid (see [1], [5], and [8]) which is applied after a time step of a usual finite-volume method. A third approach modifies the evolution equation (see [21], [6]) so that the error in the constraint is advected or diffused away. The common idea of these approaches is to correct an existing error of the constraint. See also [26] for a collection and comparison of methods and [7] for an approach on unstructured grids.

The staggered approach is equivalent to the mimetic discretizations as presented in [13] and [14] if applied to a rectangular grid. These schemes store different variables at different locations in the grid, like edges and vertices. The starting point is to derive discrete vector-analytic identities using special div-, curl-, and grad-definitions. These identities are responsible for discrete constraint preservation. The results of [13] are used in computational electrodynamics; see [14]. The application in finite-volume schemes is complicated since the usage of cell averages for the variables is then mandatory. Examples for a staggered grid scheme in meteorological flows and in vorticity methods are given in [19] and [12], respectively.

This paper will present a general framework for constructing genuine locally constraint-preserving finite-volume methods. We aim at explicit methods that use only a primary finite-volume grid. All variables will be stored in the cell centers and considered as cell mean values. The constraints as well as the flux operator will be discretized with this single grid using the cell averaged values. As an example we will consider so-called constraint-preserving advection equations. These advection equations must be seen as model equations for general evolution equations with constraints. Besides this they also provide interesting aspects of the advection of vector fields. The application of the presented general framework to constraint-preserving advection leads to an upwind method which exactly preserves the local values of the discrete constraint. This is the discrete imitation of the analytic property. The main idea is the usage of a special discrete operator for the constraint. Since the constraint and its preservation are relevant only in more than one dimension the resulting scheme is necessarily multidimensional. We obtain a method that is second order in time and space and is stable for Courant numbers $|c| \leq 1$. Consistency and stability are proven. Several numerical experiments with smooth and discontinuous solutions demonstrate the performance of the scheme. Within the framework we could also re-derive two methods that are known in the literature but which are inappropriate for solving constraint-preserving advection due to instabilities. The main part of the paper considers a two-dimensional setting on a rectangular grid. The presented framework also applies to general grids at the cost of more involved calculations. Methods on unstructured grids are the subject of future work. In the last section we give a sketch of the method in three dimensions.

The paper is organized as follows: In the next section we introduce constrained advection equations for vector fields and discuss their properties and their relation to
ordinary advection as well as to real physical models. In section 3 the general framework is presented that describes how numerical constraint-preserving methods may be constructed. The application of this framework to constraint-preserving advection follows in section 4. In the beginning of that section we discuss discrete constraint operators and deduce some instructive methods, while the final upwind method and its properties are presented in section 4.3. Section 5 is devoted to the numerical experiments and considers smooth as well as discontinuous solutions. Finally, we give details of the three-dimensional case in section 6 and draw conclusions in the last section.

2. Constrained advection equations. We consider a given velocity field $\mathbf{v}$ in a domain $\Omega$ of the three-dimensional space

$$\mathbf{v} : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$$

which remains constant in time. A second vector field $\mathbf{u} : \Omega \to \mathbb{R}^3$ is said to be advected in the velocity field $\mathbf{v}$ if it obeys the evolution equation

$$\partial_t \mathbf{u} + \text{div}(\mathbf{u} \otimes \mathbf{v}) = 0 \quad \text{in} \ \Omega,$$

where the divergence acts on the rows of the tensorial product $\mathbf{u} \otimes \mathbf{v}$, i.e., in the components of $\mathbf{v}$. Hence, (2) represents scalar advection equations for each component of $\mathbf{u}$. In components the vector field $\mathbf{u}$ and the advection velocity $\mathbf{v}$ are written as

$$\mathbf{u} = (u^{(x)}, u^{(y)}, u^{(z)})^T, \quad \mathbf{v} = (v^{(x)}, v^{(y)}, v^{(z)})^T.$$

An evolution like (2) represents a raw model for virtually any physical transport process. Correspondingly there exists a vast amount of work concerning analytical and numerical aspects of (2) in the literature. Note that advection of type (2) decouples the components of the vector field $\mathbf{u}$ and each component is advected separately. We will call this ordinary advection.

2.1. $\text{div}$ / $\text{curl}$-preserving advection. There are two more evolution equations which we shall show to be closely related to ordinary advection. They follow formally from (2) by replacing the differential operator and the tensorial product. We write

$$\begin{align*}
\partial_t \mathbf{u} + \text{grad}(\mathbf{u} \cdot \mathbf{v}) &= 0, \\
\partial_t \mathbf{u} + \text{curl}(\mathbf{u} \times \mathbf{v}) &= 0,
\end{align*}$$

where $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ denote the scalar product and the cross product, respectively. Note that the components of $\mathbf{u}$ are now coupled in the equations (4).

Since for any function $\psi$ we have $\text{curl grad} \psi \equiv 0$ and $\text{div curl} \psi \equiv 0$ we can deduce an accompanying equation for both types of evolutions in (4) which may be integrated. We obtain for the considered domain $\Omega$

$$\begin{align*}
&\text{for } (4)_{\text{grad}} \quad \text{curl } \mathbf{u} = \text{const in time}, \\
&\text{for } (4)_{\text{curl}} \quad \text{div } \mathbf{u} = \text{const in time}
\end{align*}$$

as additional equations. These equations state that the curl of the vector field in the case of $(4)_{\text{grad}}$ or its divergence in the case of $(4)_{\text{curl}}$ stays locally (hence globally) unaffected from the evolution. The initial fields of curl or divergence in the particular cases are frozen and their values stay locally the same. We therefore denote the evolution equation $(4)_{\text{grad}}$ by curl-preserving advection and $(4)_{\text{curl}}$ by div-preserving advection.
The equations (5) may be viewed as intrinsic or inherent constraints to the evolution equations in (4). In the language of [4] these constraints form involutions of the equations (4). We stress that these constraints are intrinsic to the evolution equations since they must not be additionally imposed to the solution. They are an inherent property of the transport operator. Any analytic solution of (4) fulfills the constraints of (5) automatically. However, this might not be true in a numerical setting where the equations are discretized. Furthermore, the apparently elliptic character of the constraints do not influence the character of the evolution. We will show later that the equations in (4) are purely hyperbolic.

2.2. Physical examples. Though less frequently than ordinary advection, the constraint-preserving evolution equations in (4) may be found in physical models as well. Furthermore, both equations should be viewed as special cases of more general models where the differential evolution operators act on general functions of \( u \).

A well-known example is the Maxwell equations of electrodynamics

\[
\begin{align*}
\partial_t \mathbf{B} + \text{curl} \mathbf{E} &= 0, \\
\partial_t \mathbf{D} - \text{curl} \mathbf{H} &= \mathbf{j},
\end{align*}
\]

where \( \mathbf{B} \) is the magnetic flux density and \( \mathbf{D} \) the electric displacement. Both evolutions have the structure of (4)\(_{\text{curl}}\). Since \( \text{div} \mathbf{B} = 0 \) is stated in the third Maxwell equation, the intrinsic constraint of (6)\(_1\) establishes the solenoidality of the \( \mathbf{B} \)-field during the entire evolution. The second equation (6)\(_2\) together with the fourth Maxwell equation \( \text{div} \mathbf{D} = \rho \) yields the conservation law for the charge density \( \rho \). This additional law must be viewed as a constraint to the evolution (6)\(_2\).

In ideal magnetohydrodynamics of plasma flows only the first Maxwell equation (6)\(_1\) plays a role and \( \mathbf{E} \) is given by \( \mathbf{E} = -\mathbf{v} \times \mathbf{B} \), where \( \mathbf{v} \) is the plasma velocity. Thus we have

\[
\begin{align*}
\partial_t \mathbf{B} + \text{curl}(\mathbf{B} \times \mathbf{v}) &= 0
\end{align*}
\]

as evolution equation for the \( \mathbf{B} \)-field which is identical to (4)\(_{\text{curl}}\). Due to the intrinsic constraint of (7), the divergence of \( \mathbf{B} \) remains zero if it is zero initially. Since this property is spoiled in an ordinary numerical calculation, the preservation of the divergence is a major issue in computational magnetohydrodynamics; see, e.g., [6].

The Navier–Stokes equations for incompressible flow read as

\[
\begin{align*}
\partial_t \mathbf{v} + \mathbf{v} \cdot \text{grad} \mathbf{v} + \text{grad} p &= \Delta \mathbf{v}, \\
\text{div} \mathbf{v} &= 0,
\end{align*}
\]

where \( \mathbf{v} \) is the flow velocity and \( p \) is the pressure. Note that the second equation is not an intrinsic constraint. It does not follow from the evolution equation for \( \mathbf{v} \); instead it is an equation to determine the pressure. In some approaches (see, e.g., [12]) the system of Navier–Stokes is rewritten in terms of the vorticity \( \Omega = \text{curl} \mathbf{v} \). The evolution equation for the vorticity may then be found from (8)\(_1\) using the identity \( \mathbf{v} \cdot \text{grad} \mathbf{v} = \text{grad} \frac{1}{2} \mathbf{v}^2 - \mathbf{v} \times \text{curl} \mathbf{v} \) and is given by

\[
\begin{align*}
\partial_t \Omega + \text{curl}(\Omega \times \mathbf{v}) &= \Delta \Omega.
\end{align*}
\]

This represents again a div-preserving evolution like (4)\(_{\text{curl}}\).

An evolution like (4)\(_{\text{grad}}\) appears to be less frequent. It is encountered, for example, in meteorological models where it originates from the system for shallow water
flows, and the preservation of curl $\mathbf{v}$ is a concern in numerical meteorology; see, e.g., [19]. The shallow water system is usually written in conservation laws

$$
\partial_t h + \text{div}(h\mathbf{v}) = 0,
$$

$$
\partial_t h\mathbf{v} + \text{div}(h\mathbf{v} \otimes \mathbf{v} + (\frac{1}{2} g h^2)\mathbf{I}) = 0
$$

for the water height $h$ and the flow velocity $\mathbf{v}$. The gravitational constant is $g$. In meteorology the flow is assumed to be smooth and the momentum balance (10) is reduced to an equation for $\mathbf{v}$. Using the first equation and again the identity $\mathbf{v} \cdot \text{grad} \mathbf{v} = \text{grad} \frac{1}{2} \mathbf{v}^2 - \mathbf{v} \times \text{curl} \mathbf{v}$ we obtain

$$
\partial_t \mathbf{v} + \text{grad}(\frac{1}{2} \mathbf{v}^2 + g h) = \mathbf{v} \times \Omega,
$$

where again $\Omega = \text{curl} \mathbf{v}$ is introduced. In this equation the curl-preserving operator of (4) is present. Here, the vector field $\mathbf{u}$ coincides with the advection velocity $\mathbf{v}$. The shallow water system is a two-dimensional model ($\partial_z \equiv 0$) with vanishing $z$-component of $\mathbf{v}$. Hence, the vorticity $\Omega$ has only one nonvanishing component $\Omega = \partial_x v(y) - \partial_y v(x)$ and the right-hand side of (11) has the form $(\Omega v(y), -\Omega v(x))^T$.

### 2.3. Identification as degenerated advection

We return to the equations in (4) to discuss more of its properties. So far it is not obvious that these equations are related to a kind of advection. Clearly, they state processes different from ordinary advection. We proceed to uncover the relation. A first inspection leads to the fact that both equations in (4) may be transformed into the form of a conservation law

$$
\partial_t \mathbf{u} + \text{div} f(\mathbf{u}) = 0
$$

with appropriate definition of the matrix $f(\mathbf{u})$. We obtain

$$
\begin{align*}
\text{for (4)}_{\text{grad}} & \rightarrow \partial_t \mathbf{u} + \text{div}((\mathbf{u} \cdot \mathbf{v})\mathbf{I}) = 0, \\
\text{for (4)}_{\text{curl}} & \rightarrow \partial_t \mathbf{u} + \text{div}((\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}) = 0,
\end{align*}
$$

where $\mathbf{I}$ represents the identity matrix. Thus, in both processes each component of $\mathbf{u}$ is conserved and the evolution equations are conservation laws. Now, to investigate (12) the matrix $\mathbf{A}(\mathbf{n})$ of linear combinations of the directional Jacobians of the flux function $f$ is formed. We have

$$
\mathbf{A}(\mathbf{n}) = Df(\mathbf{u})\mathbf{n},
$$

where $\mathbf{n}$ is a space direction to be chosen. This matrix is used to classify a conservation law; see, e.g., [23]. An equation is hyperbolic if $\mathbf{A}(\mathbf{n})$ has real eigenvalues and a complete set of eigenvectors for any direction of $\mathbf{n}$. The eigenvalues are then interpreted as characteristic velocities. The eigenvectors represent the part of the conserved vector $\mathbf{u}$ which is transported with the corresponding velocity.

We recall that in case of ordinary advection we have $f(\mathbf{u}) = \mathbf{u} \otimes \mathbf{v}$ and

$$
\mathbf{A}^{\text{(ordinary)}}(\mathbf{n}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{I} \rightarrow \lambda_{1,2,3} = \mathbf{n} \cdot \mathbf{v}, \quad V_{1,2,3} = \mathbb{R}^3
$$

with eigenvalues $\lambda_i$ and corresponding eigenspaces $V_i$. The eigenvalue $\mathbf{n} \cdot \mathbf{v}$ is real and threefold and the complete 3-space is the eigenspace for this eigenvalue. The process of advection may be defined by the presence of an eigenvalue $\mathbf{n} \cdot \mathbf{v}$. This follows from the Friedrichs diagram (see, e.g., [16]), which displays the propagation of a point disturbance associated to a certain characteristic velocity. In case of advection, the point disturbance remains a point and is simply propagated with the advection velocity $\mathbf{v}$. Due to the eigenspace in this case any vector from 3-space can be advected which corresponds to the decoupling of the advection equations in (2).
For the evolution equation of type $(4)_{\text{grad}}$ we obtain

\[
A^{\text{(grad)}}(n) = n \otimes v \Rightarrow \begin{cases}
\lambda_1 = n \cdot v, & V_1 = [n], \\
\lambda_{2,3} = 0, & V_{2,3} = [v]^\perp
\end{cases}
\tag{15}
\]

for the eigenvalues and eigenspaces. One eigenvalue is given by $n \cdot v$ which identifies the process as advection. However, there exists a second eigenvalue which is zero, and this leads to a splitting of the 3-space into two eigenspaces. That is, not all components of a vector $u$ are advected in the Friedrichs diagram. Indeed, according to $V_{2,3}$ any vector orthogonal to the advection velocity will stay in place. Any other vector is simply advected. This behavior must be viewed against the background of the constraint-preserving property: The eigenvalue $\lambda_{2,3}$ represents the constraint mode (see also [22]) which keeps the curl of $u$ locally untouched.

The evolution of type $(4)$ shows the analogous behavior. Eigenvalues and eigenspaces are given by

\[
A^{\text{(curl)}}(n) = (n \cdot v)I - v \otimes n \Rightarrow \begin{cases}
\lambda_1 = 0, & V_1 = [v], \\
\lambda_{2,3} = n \cdot v, & V_{2,3} = [n]^\perp
\end{cases}
\tag{16}
\]

Again we can identify the process as advection due to the eigenvalue $\lambda_{2,3}$. The first eigenvalue is zero and represents the constraint mode. In this case vectors parallel to $v$ remain untouched due to the eigenspace $V_1$. This corresponds to the preservation of the divergence of $u$.

Note that it is not possible to decouple the equations in (12) since the spatial derivatives do not diagonalize simultaneously. Furthermore, though only real eigenvalues exist, the hyperbolicity of the evolution equations in (12) degenerates due to the lack of eigenvectors in certain cases. Indeed, in both cases of the evolutions (4), directions $n$ orthogonal to the advection velocity give only $V_1 \subset V_{2,3} \subset \mathbb{R}^3$.

\subsection*{2.4. Special cases.}
It is instructive to consider some special cases of curl-preserving and div-preserving advection. They will emphasize the advection character of the equations.

If the value of the curl or of the divergence of $u$ is assumed to vanish initially in $\Omega$, i.e.,

\[
\begin{align*}
\text{for } (4)_{\text{grad}} & \rightarrow \quad \text{curl } u \equiv 0, \\
\text{for } (4)_{\text{curl}} & \rightarrow \quad \text{div } u \equiv 0,
\end{align*}
\tag{17}
\]

then their value will stay zero for all times. For div-preserving advection in case of $u$ being a magnetic flux this is the physically relevant case. If we additionally assume a constant advection velocity

\[
\text{grad } v = 0,
\tag{18}
\]

all evolution equations $(2)$, $(4)_{\text{grad}}$, and $(4)_{\text{curl}}$ are reduced to the form

\[
\partial_t u + v \cdot \text{grad } u = 0.
\tag{19}
\]

Hence, all types of advection become indistinguishable from constant advection.

If, still under the assumption of vanishing constraints, the velocity field is purely rotational, we have

\[
\text{grad } v = -(\text{grad } v)^T.
\tag{20}
\]
In such a case the advection given in (4) differs from ordinary advection. However, if we consider the 2-norm of \( \mathbf{u} \), we obtain the ordinary advection equation

\[
\partial_t \|\mathbf{u}\|^2 + \mathbf{v} \cdot \text{grad} \|\mathbf{u}\|^2 = 0
\]

for both curl-preserving and div-preserving advection. Hence, \( \|\mathbf{u}\|^2 \) is rotated as scalar quantity. The components of \( \mathbf{u} \), however, are not advected as scalar quantities. In fact, the vector \( \mathbf{u} \) is advected as a whole preserving its position relative to the rotating velocity.

2.5. Two-dimensional equations. The numerical methods in the next sections are mainly developed for the two-dimensional case, that is, \( \partial_z \to 0 \). We proceed to display the two-dimensional equations.

For the div-preserving advection (4) the equation for the component \( u(z) \) decouples from the first two equations for \( (u(x), u(y)) \). Furthermore, the constraint \( \text{div} \mathbf{u} \) is no longer influenced by \( u(z) \). Hence, we will discard the equation for \( u(z) \) in the following. The remaining equations are given by

\[
\begin{align*}
\partial_t u(x) + \partial_y (u(x) u(y) - v(x) u(y)) &= 0, \\
\partial_t u(y) - \partial_x (u(x) u(y) - v(x) u(y)) &= 0
\end{align*}
\]

for the components \( (u(x), u(y)) \). Note that \( u(z) \) is not zero, nor is its evolution trivial. The component \( u(z) \) and its evolution simply does not play a role in the following construction of div-preserving methods.

In the two-dimensional case of (4) it follows \( u(z) = \text{const} \) in time; however, the equations for \( (u(x), u(y)) \) still depend on \( u(z) \). In the most important application of curl-preserving advection—the shallow water system—the additional condition \( v(z) = 0 \) holds, which yields decoupled equations. Having in mind this kind of application, the remaining equations for the components \( (u(x), u(y)) \) are written as

\[
\begin{align*}
\partial_t u(x) + \partial_y (u(x) v(x) + v(y) u(y)) &= 0, \\
\partial_t u(y) + \partial_x (u(x) v(x) + v(y) u(y)) &= 0
\end{align*}
\]

for the two-dimensional version of (4). As in the case of (22) the component \( u(z) \) is not further considered.

Correspondingly the constraints are written

\[
\begin{align*}
\text{for (4) grad } \to & \quad \partial_x u(y) - \partial_y u(x) = \text{const}, \\
\text{for (4) curl } \to & \quad \partial_x u(z) + \partial_y u(y) = \text{const}
\end{align*}
\]

in two dimensions. Note that the constraint of type (4) grad, which was a vectorial quantity in (5), became a scalar equation.

The dual behavior of curl-preserving and div-preserving advection already observable in the previous section becomes perfect in the two-dimensional case. By substituting the two-dimensional vector \( \mathbf{u} \) by its orthogonal complement

\[
\begin{pmatrix} u(x) \\ u(y) \\ u(y) \\ -u(x) \end{pmatrix} \leftrightarrow \begin{pmatrix} u(x) \\ u(y) \\ -u(x) \\ u(y) \end{pmatrix}
\]

we can transform curl-preserving and div-preserving advection into each other. Hence, in what follows, any statement or numerical method for the system (23) can be transformed into a statement or numerical method for (22) with the same properties and vice versa.
3. General framework. A numerical solution of equations like curl-preserving and div-preserving advection should respect the intrinsic constraints in a numerical way. That is, a discrete version of the constraint should follow directly from the numerical discretization of the evolutions. Ordinary numerical schemes, however, do not care about the constraints which leads to well-known problems, e.g., in calculating magnetohydrodynamical flows [6]. We propose that a numerical scheme has to be constructed on the basis given by a discretization of the constraints. Since the equations of interest are hyperbolic with local domain of dependency we expect that the constraint can be controlled locally as well. In this section we set up a general framework for locally constrained transport schemes.

We consider \( u \in \Omega \subseteq \mathbb{R}^D \) \((D: \text{space-dimension})\) and a generic evolution

\[
\partial_t u + F(u; x) = 0
\]

with a transport operator \( F \) depending explicitly on the space variable \( x \). The generic constraint \( C \) is assumed to be linear and intrinsic for (26), that is, the relation

\[
C(F(u; x)) \equiv 0
\]

holds independently of \( u \) and \( x \), which directly implies

\[
C(u) = \text{const in time}
\]

for any solution of (26); see also [4]. The computational domain \( \Omega \) is covered by a grid \( T = \{K_i\}_{i=1,2,...} \) with nonoverlapping polygonal cells \( K_i \) whose inner diameter is bounded by \( h \). Two cells are called neighbors if they have a common edge or if they only share a vertex. The set \( \mathcal{N}(K) \) gives all neighbors of the cell \( K \). A time discretization by \( \Delta t \) leads to a cell-wise constant grid function \( \tilde{u}^m : T \to \mathbb{R}^D \) which approximates \( u \) after \( m \) time steps by cell mean values.

3.1. Flux distribution formulation. The central quantity of this paper is the so-called flux distribution. It will be the structure of the flux distribution that determines whether a certain scheme is constraint-preserving.

**Definition 3.1 (flux distribution).** Given the space of vector-valued grid functions denoted by \( V = \{g : T \to \mathbb{R}^D\} \), we define a “flux distribution” \( \Phi_K : V \to V \) which is attached to a grid cell and maps the grid function \( \tilde{u} \) into another grid function, that is,

\[
\Phi_K(\tilde{u}) : T \to \mathbb{R}^D
\]

with \( \text{supp}(\Phi_K(\tilde{u})) = K \cup \bigcup_{K \in \mathcal{N}(K)} \hat{K} \). The evaluation \( \Phi_K(\tilde{u})|_K \) gives the change of \( \tilde{u} \) at cell \( K \) caused by cell \( K \) during a time step, that is, the flux.

A flux distribution is assigned to each cell of the grid and may depend on the value of \( \tilde{u} \) in this particular cell but also on that in other cells. The definition is more general than that of usual intercell fluxes, since it admits fluxes to any neighboring cell, especially across corners. This incorporates multidimensionality from the very beginning. Conservation of \( \tilde{u} \) may be expressed by the statement that the integral of \( \Phi_K(\tilde{u}) \) vanishes.

A certain form of the flux distribution and its dependency on \( \tilde{u} \) is usually constructed from consistency with the transport equation. Once the flux distribution is defined, an explicit evolution scheme follows by simply collecting contributions of all
flux distributions. Written in complete grid functions we have

\[ \tilde{u}^{m+1} = \tilde{u}^m + \sum_K \Phi_K (\tilde{u}^m) \]

as an update for the complete grid. The restriction to a certain cell yields a local formulation, viz.,

\[ \tilde{u}^{m+1}|_K = \tilde{u}^m|_K + \sum_{K \in \{K \cup N(K) \}} \Phi_K (\tilde{u}^m)|_K. \]

Here the value of \( \tilde{u} \) in a cell \( K \) is updated by contributions of all neighboring cells which are given by evaluations of flux distributions. Note that virtually any finite-volume scheme can be written in the form (31), and the flux distribution may then be identified.

To approximate the transport equation given in (26) consistency is required in the form of cell mean values, viz.,

\[ F(u; x)|_K = -\lim_{\Delta t, h \to 0} \frac{1}{\Delta t} \sum_{K \in \{K \cup N(K) \}} \Phi_K (\tilde{u})|_K, \]

where \( \tilde{u} \) is the projection of \( u \) onto the grid.

In order to clarify the notion of a flux distribution, we briefly present a possible flux distribution for one-dimensional constant advection \( u_t + a u_x = 0 \) on a uniform mesh. The flux distribution given by

\[ \Phi_i (\tilde{u}^m) = \begin{cases} \max(0, a) \Delta t \tilde{u}_i^m & \text{for cell } i + 1, \\ -\left( \max(0, a) \Delta t + \min(0, a) \Delta t \right) \tilde{u}_i^m & \text{for cell } i, \\ \min(0, a) \Delta t \tilde{u}_i^m & \text{for cell } i - 1 \end{cases} \]

has entries in cell \( i \) and its neighbors \( i \pm 1 \). The evolution (31) may then be written

\[ \tilde{u}^{m+1}_i = \tilde{u}_i^m - \frac{\max(0, a) \Delta t}{h} (\tilde{u}_i^m - \tilde{u}_{i-1}^m) + \frac{\min(0, a) \Delta t}{h} (\tilde{u}_{i+1}^m - \tilde{u}_i^m), \]

which represents the donor cell scheme for constant advection.

### 3.2. Constraint preservation.

Since the constraint is linear we expect a discretization which may be written as matrix operation

\[ \mathcal{C}(u)|_K = \hat{C}_K \tilde{u} + O(h^n) \]

on the grid function \( \tilde{u} \) which is obtained from the function \( u \) by cell-wise constant projection. If preservation of the constraint should be achieved for the scheme (30), the following quite obvious statement leads the way.

**Lemma 3.2 (constraint preservation).** If the condition

\[ \hat{C}_K \Phi_K (\tilde{u}) = 0 \quad \forall K, \hat{K}, \tilde{u} \]

holds for a specific discrete constraint and a flux distribution, it follows for the evolution scheme given in (30)

\[ \hat{C}_K \tilde{u}^{m+1} = \hat{C}_K \tilde{u}^m, \]

i.e., the discrete operator is preserved locally by this scheme.
Note that condition (36) is sufficient only for constraint preservation, since the flux distribution is completely unspecified. Contributions of different flux distributions in (30) could interact in such a way that the discrete constraint is preserved even if (36) is not valid. However, we will not consider such schemes.

Since the condition is difficult to control for any grid function \( \tilde{u} \), we assume a decomposition

\[
\Phi_K (\tilde{u}) = \varphi_K (\tilde{u}) \hat{\Phi}_K
\]

into a factor \( \varphi_K (\tilde{u}) \in \mathbb{R} \) and a skeleton or shape function \( \hat{\Phi}_K \). As indicated, only the factor depends on the field \( \tilde{u} \). Due to the linearity of the constraint this factor drops out of the condition in (36) and we obtain

\[
\tilde{C}_K \hat{\Phi}_K = 0 \quad \forall K, \hat{K}
\]

as a purely geometric condition. To some extent this is the discrete analogon to (27) which states that the constraint is intrinsic. Indeed, for the case of div-preserving advection the curl in (4) must be discretized in an update \( \hat{\Phi}_K \) such that a discrete divergence \( \hat{C}_K \) gives exactly zero. This is also the approach in [13], [14], where discrete analogons of vector-analytic relations are considered and used to discretize Maxwell’s equations. The work of [13], [14], however, relies on using different locations, i.e., cell-center, face, edge, and vertex, to discretize vector functions and define the differential operators. The operators div and curl are then defined on different grids and for differently stored variables. For a finite-volume approach with exclusive use of cell mean values this is unsatisfactory. The condition in (39) aims at discrete operators and updates that use only cell centered variables. However, at least one of the resulting schemes may be translated in a “mimetic” scheme described in [14] by appropriate averaging. This will be demonstrated at the end of section 4.2.2.

If a generic cell \( \hat{K} \) is fixed, (39) gives a homogeneous linear system of equations and the flux distributions are elements of its kernel. The system will be finite if the discrete constraint has a finite stencil since then evaluations of (39) for cells \( K \) far off the support of \( \hat{\Phi}_K \) will vanish identically. The crucial task of designing constrained transport schemes is to find nontrivial solutions of (39) for a given discrete constraint operator. A nontrivial solution of (39) is expected to exist if functions with compact support exist for which the analytic constraint vanishes. The structure of the solutions depends strongly on the discretization used of the constraint operator.

The system (39) for a fixed cell \( \hat{K} \) is homogeneous and possesses more equations than unknowns since the evaluation of \( \hat{C}_K \) on cells neighboring the support of \( \hat{\Phi}_K \) will yield nontrivial expressions. Experience with concrete examples showed that due to symmetry most equations are linear dependent and the entire system has a rank less than the number of unknowns. However, a proof of the general statement that the system (39) always has a rank less than the number of equations is not yet available. We expect a solution space for (39) from which we only consider an appropriate basis set of flux distributions \( \{ \hat{\Phi}_K^{(g)} \} \) with \( g = 1, 2, \ldots \) which are all constraint-preserving.

The final flux distribution has to be assembled from these solutions,

\[
\Phi_K (\tilde{u}) = \sum_g \varphi_K^{(g)} (\tilde{u}) \hat{\Phi}_K^{(g)}
\]

with unknown coefficients \( \varphi_K^{(g)} \). Note that the choice of \( \varphi_K^{(g)} \) does not affect the preservation of the constraint which is already established by the skeletons of the
flux distributions. The expression for $\Phi_K$ enters the scheme (31), and the remaining coefficients $\varphi^{(g)}_K$ have to follow from consistency (32) and stability as well.

The local character of the constraint is crucial at this point. If the constraint has a global influence, like the divergence condition in the elliptic Stokes problem, it will not be possible to find a flux distribution which is consistent and locally constraint-preserving. In the case of the elliptic Stokes problem either the constraint condition (39) for a consistent flux distribution or the consistency condition (32) for a preserving one would result in a global problem accounting for the ellipticity.

4. Rectangular grid in two dimensions. We proceed with applying the general framework of the preceding section to the system (22), thus concentrating on div-preserving advection. Both equations in (22) are governed by a single flux function which we denote by

$$F(u, v) = u^{(x)}v^{(y)} - v^{(x)}u^{(y)}.$$ \hfill (41)

As indicated in section 2.3, a numerical scheme for (22) can be directly transformed into a scheme for (23) by duality. Further investigations will be presented in the case of a rectangular grid with cells $K = (i, j)$ at positions $(x_i, y_j)$ and size $\Delta x \times \Delta y$. The geometry factor of the grid $\alpha = \frac{\Delta x}{\Delta y}$ shall be bounded from above and below. In cases of accuracy considerations we refer to $h = \max(\Delta x, \Delta y)$.

Note that the general framework is valid for any kind of polygonal grid. However, the construction and investigation of discrete constraint operators on triangular or quadrilateral grids become complicated. The extension to more general grids is subject to future work.

4.1. Discrete constraints. Since the discrete version of the constraint operator influences the structure of the flux distribution, we present a certain class of discrete divergence operators. Each operator is obtained from a discretization of the first derivative in a finite difference approach. We require a symmetric $3 \times 3$ stencil and an approximation of second order. The following lemma gives all possible approximations of this type.

**Lemma 4.1 (discrete first derivatives).** Any second order approximation of the first derivative of a smooth function $\psi$ in the center cell $(i, j)$ of a symmetric $3 \times 3$ Cartesian stencil has the form

$$\left. \frac{\partial \psi}{\partial x} \right|_{i,j} = \tilde{D}_{i,j} (\alpha, \beta, \gamma) \tilde{\psi} + O(h^2)$$ \hfill (42)

with arbitrary values for $\alpha, \beta, \gamma$ and

$$\tilde{D}_{i,j} (\alpha, \beta, \gamma) = \frac{1}{2\Delta x} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha}{\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$+ \frac{\beta}{\Delta x} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix} + \frac{\gamma}{\Delta x} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$ \hfill (43)

as weights in the grid. In the tables the middle cell corresponds to the cell $(i, j)$. 
Proof. In the case \( \alpha = \beta = \gamma = 0 \) the operator (43) reduces to the classical symmetric finite differences which are visible in the first block of (43). Assuming sufficient smoothness of \( \psi \), this gives a second order approximation to the first derivative of \( \psi \). We need only to show that the additional blocks in (43) do contribute only terms of \( O(h^2) \). Indeed, since these blocks are discretizations of higher order cross derivatives, evaluation yields

\[
\tilde{D}(\alpha, \beta, \gamma) \psi = \frac{\partial \psi}{\partial x} + \frac{\alpha}{\Delta x} \left( 2\Delta x \Delta y^2 \frac{\partial^3 \psi}{\partial x^2 \partial y^2} + O(h^3) \right)
+ \frac{\beta}{\Delta x} \left( 2\Delta x^2 \Delta y \frac{\partial^3 \psi}{\partial x^2 \partial y} + O(h^3) \right)
+ \frac{\gamma}{\Delta x} \left( \Delta x^2 \Delta y^2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + O(h^4) \right)
= \frac{\partial \psi}{\partial x} + O(h^2).
\]

Finally, we observe that it is not possible to include more discrete cross derivatives. They would be built from at least third order derivatives with respect to \( x \) or \( y \) which have no representation with a \( 3 \times 3 \) stencil.

In the following, we will evaluate discrete derivatives via (43) by using cell mean values instead of point values, which introduces only an error of second order.

The constraint for (4) for curl in (24) is now to be replaced with a discrete formulation. By the lemma on discrete first derivatives given above a discrete divergence operator \( \tilde{C}_{i,j} \) has the form

\[
\text{div} \, \mathbf{u}|_{i,j} = \tilde{D}_{i,j} (\alpha, \beta, \gamma) \tilde{u}^{(x)} - \tilde{D}^T_{i,j} (\alpha, \beta, \gamma) \tilde{u}^{(y)} + O(h^2)
\]

(44)

which leads to a three parameter family of operators. We mention that if any operator taken from (44) by specifying \( \alpha, \beta, \gamma \) vanishes, all other operators give a result of \( O(h^2) \) for smooth functions. Hence, if a numerical scheme respects one operator exactly, all others will give only a discretization error. Even in the discontinuous case the control of a single operator is sufficient to avoid nonphysical solutions. See, e.g., [5], where a staggered operator is controlled.

4.2. Flux distributions. In order to derive divergence-preserving flux distributions we have to look for nontrivial solutions of (39) for a specific operator chosen from (44). The flux distribution skeleton \( \tilde{\Phi}_{i,j} \) in the two-dimensional rectangular case covers a region of \( 3 \times 3 \) cells and gives a two-vector in each cell, thus consisting of \( 2 \times 9 = 18 \) unknown entries. If we fix \( K \) in (39) and evaluate the divergence operator around the flux distribution in the cells of a \( 5 \times 5 \) area with center \( K \), we obtain a system of 25 equations which describes the skeleton entries. The divergence of more remote cells is not influenced by the flux distribution at cell \( K \) and need not to be considered. The resulting system, of course, depends on the chosen operator.

4.2.1. Classical operator. The classical discrete divergence operator \( \tilde{C}_{i,j}^{(0)} \) is obtained from (44) by setting \( \alpha = \beta = \gamma = 0 \), which leads to

\[
\tilde{C}_{i,j}^{(0)} \tilde{u} = \tilde{C}_{i,j} (0, 0, 0) \tilde{u} = \frac{\tilde{u}_{i+1,j}^{(x)} - \tilde{u}_{i-1,j}^{(x)}}{2\Delta x} + \frac{\tilde{u}_{i,j+1}^{(y)} - \tilde{u}_{i,j-1}^{(y)}}{2\Delta y}.
\]

(45)
Fig. 1. Several shape functions for flux distributions that are divergence-preserving. All of them share the same physical interpretation: They have to approximate closed curves in order to avoid introducing sources into the vector field $u$ and thus they preserve the divergence. In terms of differential forms, these flux distributions are minimal co-cycles of the discrete outer derivative.

In general, this operator represents the best second order approximation to $\text{div} u$ in the sense that the constant hidden in $O(h^2)$, i.e., the residual of the second order Taylor expansion, is minimal. The linear system (39) for this operator has rank 17, thus it has a one-dimensional null space. We choose a representative of this kernel and denote it by $\hat{\Phi}^{(0)}_{i,j}$. The nonvanishing entries are given by

$$\hat{\Phi}^{(0)}_{i,j} \big| \begin{array}{c} i+1,j \\ i,j+1 \\ i,j-1 \\ i-1,j \end{array} = \begin{cases} (0, \Delta y), & \hat{\Phi}^{(0)}_{i,j} \big| i,j+1 = (-\Delta x, 0), \\ \Delta x, 0, & \hat{\Phi}^{(0)}_{i,j} \big| i,j-1 = (0, -\Delta y), \\ 0, \Delta y, & \hat{\Phi}^{(0)}_{i,j} \big| i-1,j = \Delta x, 0, \\ \end{cases}$$

and all other elements of the kernel follow by multiplication with a constant factor. This flux distribution is sketched in the upper left corner of Figure 1. The picture has to be interpreted as follows: A flux originating in $K$ may change the value of $\tilde{u}$ in the right neighboring cell only in the $y$-direction. Furthermore, if this neighbor is changed in that way, all other neighboring cells have to be changed correspondingly as depicted in the figure in order to obey the constraint. Of course, the value $\tilde{u}_{x}^{(x)}$ in the right neighbor of $K$ does not remain constant, since it may be changed by flux distributions originating from other cells. This kind of flux distribution results in a coupling of fluxes into neighboring cells, and it is this coupling that is responsible for the local preservation of the divergence.

The divergence-preserving flux distributions also have interpretations in the theory of differential forms. They represent minimal discrete co-cycles of the corresponding discrete outer derivatives; see [7], [15], [24].

In order to construct the final scheme we assemble the shape function $\hat{\Phi}_{i,j}$ of the flux distribution according to

$$\Phi^{(0)}_{i,j} (\tilde{u}) = \varphi_{i,j}^{(0)} (\tilde{u}) \hat{\Phi}^{(0)}_{i,j}$$

with an unknown function $\varphi$. Note that $\hat{\Phi}_{i,j} = O(h)$ and, since $\Phi_{i,j}$ has to be $O(1)$, it follows that $\varphi_{i,j} = O(h^{-1})$. The final scheme is obtained by following (31) with (46)
and reads as

\[(\tilde{u}^{(x)})_{i,j}^{m+1} = (\tilde{u}^{(x)})_{i,j}^{m} + \left( \frac{\varphi^{(0)}_{i,j-1}(\tilde{u}) - \varphi^{(0)}_{i,j+1}(\tilde{u})}{\Delta x} \right) \Delta x_{ij} \]  

By Taylor expansion and comparison with the original equation (22), we deduce

\[\varphi^{(0)}_{i,j}(\tilde{u}) = -\frac{\Delta t}{2\Delta x\Delta y} F(\tilde{u}_{ij}, v_{ij}),\]  

which makes (48) consistent up to second order. This scheme solves for div-preserving advection while exactly preserving the value of the classical divergence operator (45). The scheme introduces central differences for the derivatives in (22) and was proposed ad hoc by Toth [26] in a magnetohydrodynamic setting. Since \(\hat{\Phi}^{(0)}_{ij}\) is the only flux distribution respecting condition (39) with the classical operator, we conclude that this scheme is the only second order scheme which preserves the divergence via the classical operator (45).

However, the scheme (48) is unconditionally unstable due to central differences. For the investigation of stability we have to look for the maximal spectral radius of the amplifier matrix

\[\rho_{\text{max}} = \max_{\xi,\eta \in (-\pi, \pi)} \rho(T_{\xi,\eta})\]

(see section 4.3.2 for more details). Assuming a constant advection velocity and defining the Courant numbers

\[a = \frac{\Delta t v^{(x)}}{\Delta x}, \quad b = \frac{\Delta t v^{(y)}}{\Delta y}\]

we obtain the result

\[\rho^{(0)}_{\text{max}} = \max_{\xi,\eta \in (-\pi, \pi)} |1 - i(a \sin \xi + b \sin \eta)| > 1\]

unless \(a = b = 0\) for the case of (48). The imaginary unit is denoted by \(i = \sqrt{-1}\). In spite of this instability the scheme (48) could be used in [26] in the context of magnetohydrodynamics due to the use of predictor values for \(u\).

4.2.2. Extended operator. In order to design a more flexible scheme we look for a different divergence operator which leads to a larger null space of (39) and thus provides more nontrivial solutions. Empirical evaluations of the family given in (43) by computer algebra software suggest that the choice \(\alpha = \frac{1}{8}, \gamma = 0\) admits a four-dimensional kernel for any value of \(\beta\); i.e., the system (39) has rank 14. The best approximation is then given by \(\beta = 0\), and the resulting operator is called expanded operator \(\tilde{C}^{(x)}_{i,j}\). Operators with larger kernels could not be found. The extended operator is defined by

\[\tilde{C}^{(x)}_{i,j} \tilde{u} = \tilde{C}_{i,j} \left( \frac{1}{8}, 0, 0 \right) \tilde{u}
\]

\[= \frac{\{\tilde{u}^{(x)}_{i+1,j}\} - \{\tilde{u}^{(x)}_{i-1,j}\}}{2\Delta x} + \frac{\{\tilde{u}^{(y)}_{i,j+1}\} - \{\tilde{u}^{(y)}_{i,j-1}\}}{2\Delta y},\]  

(53)
where curled brackets stand for
\[
\{ \psi_{i,j} \}_y = \frac{1}{4} (\psi_{i,j+1} + 2\psi_{i,j} + \psi_{i,j-1}), \quad \{ \psi_{i,j} \}_x = \frac{1}{4} (\psi_{i+1,j} + 2\psi_{i,j} + \psi_{i-1,j}),
\]
i.e., averaging in \(x\)- and \(y\)-direction. The four admissible skeletons of flux distributions are displayed in the lower row of Figure 1. In detail, the nonvanishing entries of the first one are given by
\[
\hat{\Phi}^{(1)}_{i,j} |_{i,j+1} = (\Delta x, \Delta y), \quad \hat{\Phi}^{(1)}_{i,j} |_{i,j} = (-\Delta x, -\Delta y), \quad \hat{\Phi}^{(1)}_{i+1,j} |_{i,j} = (\Delta x, -\Delta y), \quad \hat{\Phi}^{(1)}_{i,j+1} |_{i,j} = (\Delta x, \Delta y)
\]
and the remaining three flux distributions follow by translation. Note that the classical operator applied to these flux distributions will not vanish. We remark further that any scheme built upon these flux distribution skeletons will be conservative, since the cell-wise sum of all flux distribution components, i.e., the integral, gives zero.

As first choice for a flux distribution, we choose the symmetric distribution \(\hat{\Phi}^{(*)}_{i,j}\) which is given by
\[
\hat{\Phi}^{(*)}_{i,j} = \hat{\Phi}^{(1)}_{i,j} + \hat{\Phi}^{(2)}_{i,j} + \hat{\Phi}^{(3)}_{i,j} + \hat{\Phi}^{(4)}_{i,j}
\]
and shown in the upper right corner of Figure 1. Like in the preceding section this flux distribution is assembled with an unknown function \(\varphi\) to give
\[
\Phi^{(*)}_{i,j} (\tilde{u}) = \varphi^{(*)}_{i,j} (\tilde{u}) \hat{\Phi}^{(*)}_{i,j}.
\]
For the resulting scheme we obtain
\[
(\tilde{u}^{(x)})_{i,j}^{m+1} = (\tilde{u}^{(x)})_{i,j}^m + \left( \left( \left\{ \varphi^{(*)}_{i,j-1}(\tilde{u}) \right\}_x - \left\{ \varphi^{(*)}_{i,j+1}(\tilde{u}) \right\}_x \right) \Delta x \right)^m,
\]
where again the curled brackets denote the averaging of (54). The demand for second order consistency with (22) leads to
\[
\varphi^{(*)}_{i,j} (\tilde{u}) = -\frac{\Delta t}{2\Delta x \Delta y} F (\tilde{u}_{i,j}, v_{i,j}).
\]
This scheme exactly preserves the value of the extended divergence operator (53). However, as in the case of the \(\Phi^{(0)}\)-scheme, this scheme is unconditionally unstable. For the maximal spectral radius of the amplifier matrix we calculate with constant advection and Courant numbers from (51)
\[
\rho^{(*)}_{\text{max}} = \max_{\xi,\eta \in (-\pi,\pi]} |1 - i \left( a \sin \xi \frac{1+\cos \eta}{2} + b \sin \eta \frac{1+\cos \xi}{2} \right)| \geq 1,
\]
where equality holds only if \(a = b = 0\). The instability of schemes (58) as well as (48) could also be observed in our numerical experiments.

**Equivalence with staggered approach.** In the context of magnetohydrodynamics one approach of controlling the divergence is to store the components of \(u\) in the edges of the cells, the so-called staggered grid. This idea was proposed in [8] and
further developed by [5], [1]. We follow the presentation in [26]. In our notation the resulting staggered grid scheme reads

$$
\left(\begin{array}{c}
\tilde{u}^{(x)}_{i+\frac{1}{2},j} \\
\tilde{u}^{(y)}_{i,j+\frac{1}{2}}
\end{array}\right)^{m+1} = \left(\begin{array}{c}
\tilde{u}^{(x)}_{i+\frac{1}{2},j} \\
\tilde{u}^{(y)}_{i,j+\frac{1}{2}}
\end{array}\right)^m + \left(\begin{array}{c}
\frac{\Delta t}{\Delta x} (F_{i+\frac{1}{2},j+\frac{1}{2}} - F_{i+\frac{1}{2},j-rac{1}{2}}) \\
\frac{\Delta t}{\Delta y} (F_{i+\frac{1}{2},j-rac{1}{2}} - F_{i+\frac{1}{2},j+\frac{1}{2}})
\end{array}\right)
$$

for the normal components $\tilde{u}^{(x)}_{i+\frac{1}{2},j}$ and $\tilde{u}^{(y)}_{i,j+\frac{1}{2}}$ on each edge and the function $F_{i+\frac{1}{2},j+\frac{1}{2}}$ evaluated at the vertices. This scheme corresponds to the so-called mimetic discretization of [13], [14] for the equation (4) if it is applied in the case of a rectangular grid.

In the context of computational magnetohydrodynamics and in this paper all variables are represented as mean values located in the cell centers. Hence, the staggered variables in (61) have to be substituted; see [26]. The flux function at the vertices is obtained by averaging

$$
F_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{4} (F_{i,j} + F_{i+1,j} + F_{i,j+1} + F_{i+1,j+1}),
$$

where the expression $F_{i,j}$ corresponds to the evaluation of the flux function (41) in the cell $(i,j)$. Finally, the edge values of $u^{(x)}$ and $u^{(y)}$ are averaged via

$$
\tilde{u}^{(x)}_{i,j} = \frac{1}{2} \left( \tilde{u}^{(x)}_{i+\frac{1}{2},j} + \tilde{u}^{(x)}_{i-\frac{1}{2},j} \right), \quad \tilde{u}^{(y)}_{i,j} = \frac{1}{2} \left( \tilde{u}^{(y)}_{i,j+\frac{1}{2}} + \tilde{u}^{(y)}_{i,j-\frac{1}{2}} \right)
$$

after each time step (61). In [26] it was noted that this averaging procedure can be done explicitly with the scheme (61), which leads to a scheme where staggered values of $u$ are eliminated. The resulting scheme (formerly staggered) is equivalent to the symmetric $\Phi^{(x)}$-scheme in (58).

Furthermore, this shows that the extended operator $\tilde{\Phi}^{(x)}_{i,j}$ is exactly preserved on the primary grid cells in a staggered grid calculation. This also suggests a close relation between the present extended divergence operator $\tilde{\Phi}^{(x)}_{i,j}$ and the $DIV$-operator which is preserved in the mimetic schemes of [14]. The $DIV$-operator for the normal components on the edges is written

$$
DIV \, u \big|_{i,j} = \frac{\tilde{u}^{(x)}_{i+\frac{1}{2},j} - \tilde{u}^{(x)}_{i-\frac{1}{2},j}}{\Delta x} - \frac{\tilde{u}^{(y)}_{i,j+\frac{1}{2}} - \tilde{u}^{(y)}_{i,j-\frac{1}{2}}}{\Delta y}
$$

in the rectangular case; see [13]. This is exactly the $\tilde{\Phi}^{(x)}_{i,j}$-operator if the edge values are obtained from the cell centered variables by averaging

$$
\tilde{u}^{(x)}_{i+\frac{1}{2},j} = \frac{\tilde{u}_{i+1,j}^{(x)} + 2\tilde{u}_{i+1,j}^{(x)} + \tilde{u}_{i+1,j-1}^{(x)} + \tilde{u}_{i,j+1}^{(x)} + 2\tilde{u}_{i,j}^{(x)} + \tilde{u}_{i,j-1}^{(x)}}{8},
$$

and analogously for $\tilde{u}^{(y)}_{i,j+\frac{1}{2}}$. This is also the same averaging formula which is used in [19] to switch from cell centered variables to normal edge components.

The symmetric scheme in (58) as well as the staggered approach is unstable, since these schemes do not take upwind directions into account. As in the case of (48) the staggered grid scheme is stabilized in magnetohydrodynamics calculations by use of predictors; see [1], [5], and [26].
4.3. Upwind scheme. The symmetric flux distribution (56) uses the same coefficient \( \varphi^{(*)} \) for all basis elements \( \Phi^{(g)} \). This results in central differences and instability of the final scheme. To construct an upwind scheme we propose

\[
\Phi^{(up)}_{i,j}(\tilde{u}) = \sum_{g=1}^{4} \varphi^{(g)}_{i,j}(\tilde{u}) \hat{\Phi}^{(g)}_{i,j}
\]

as flux distribution with four coefficients \( \varphi^{(g)} \) yet to be specified. The final scheme reads as

\[
\begin{align*}
\left( \begin{array}{c}
\tilde{u}^{(x)}_i \\
\tilde{u}^{(y)}_j
\end{array} \right)_{i,j}^{m+1} &= \left( \begin{array}{c}
\tilde{u}^{(x)}_i \\
\tilde{u}^{(y)}_j
\end{array} \right)_{i,j}^m + \left( \begin{array}{c}
\delta^{(y)}_{i+\frac{1}{2},j-\frac{1}{2}}(\varphi^{(1)}) \Delta x \\
-\delta^{(y)}_{i-\frac{1}{2},j-\frac{1}{2}}(\varphi^{(1)}) \Delta y
\end{array} \right) + \left( \begin{array}{c}
\delta^{(y)}_{i+\frac{1}{2},j-\frac{1}{2}}(\varphi^{(2)}) \Delta x \\
-\delta^{(y)}_{i+\frac{1}{2},j-\frac{1}{2}}(\varphi^{(2)}) \Delta y
\end{array} \right) \\
&+ \left( \begin{array}{c}
\delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2}}(\varphi^{(3)}) \Delta x \\
-\delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2}}(\varphi^{(3)}) \Delta y
\end{array} \right) + \left( \begin{array}{c}
\delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2}}(\varphi^{(4)}) \Delta x \\
-\delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2}}(\varphi^{(4)}) \Delta y
\end{array} \right),
\end{align*}
\]

where we used the abbreviations

\[
\begin{align*}
\delta_{i+\frac{1}{2},j+\frac{1}{2}}^{(y)}(\varphi) &= \varphi_{i+1,j+1} + \varphi_{i+1,j-1} - \varphi_{i,j+1} - \varphi_{i,j-1}, \\
\delta_{i+\frac{1}{2},j+\frac{1}{2}}^{(y)}(\varphi) &= \varphi_{i+1,j+1} + \varphi_{i+1,j-1} - \varphi_{i,j+1} - \varphi_{i,j-1},
\end{align*}
\]

which represent finite differences. The next two subsections specify the coefficients \( \varphi^{(g)} \) by requiring consistency and stability of the scheme (67).

4.3.1. Consistency. The following lemma gives expressions for \( \varphi^{(g)} \) such that a first or second order method is obtained. A weight function \( \omega^{(g)} \) which controls the influence of the different basis flux distributions remains unspecified.

**Theorem 4.2 (consistency).** Let the values \( \omega^{(g)} \) \( (g = 1, 2, 3, 4) \) be weights such that \( \sum_{g=1}^{4} \omega^{(g)} = 1 \). In general these weights depend on \( \tilde{u} \) and \( \nu \). Furthermore let the expressions \( \frac{\Delta t}{\Delta x} \) and \( \frac{\Delta t}{\Delta y} \) be \( O(1) \), the geometry factor \( \alpha = \frac{\Delta x}{\Delta y} \) be bounded from above and below, and \( h = \max(\Delta x, \Delta y) \). Then the scheme displayed in (67) is consistent with the constrained advection equation (22) in smooth regions of the solution up to

(i) first order in space and time, if the flux distribution factor in (66) is given by

\[
\varphi^{(g)}_{i,j}(\tilde{u}) = -\frac{\Delta t}{2\Delta x \Delta y} \omega^{(g)}(\tilde{u}_{i,j}, \nu_{i,j}) F(\tilde{u}_{i,j}, \nu_{i,j});
\]

(ii) second order in space and time, if the flux distribution factor in (66) is given by

\[
\varphi^{(g)}_{i,j}(\tilde{u}) = -\frac{\Delta t}{2\Delta x \Delta y} \omega^{(g)}(\tilde{u}_{i,j}, \nu_{i,j}) \left( F - \frac{\Delta t}{2} (v^{(x)} \partial_x F + v^{(y)} \partial_y F) + \Lambda \right)_{i,j}
\]

with \( \Lambda = \sum_{g=1}^{4} \frac{\Delta t}{2} r_g \partial_x (\omega^{(g)} F) + \frac{\Delta t}{2} l_g \partial_y (\omega^{(g)} F) \) and coefficients \( r_g \) and \( l_g \) as given in the table (71).

We remark that the second order result (70) uses derivatives of the weight function \( \omega \). Hence, \( \omega \) considered as a function in the domain \( \Omega \) needs to have at least one continuous derivative in order to obtain second order accuracy. We will present numerical experiments where second order is not recovered due to a nonsmooth weight function.
Proof. We consider the flux distribution factor \( \varphi_{i,j} \) as function \( \varphi_{i,j} \equiv \varphi(x_i, y_j) \) whose evaluations in different grid cells can be expanded in a Taylor series. Second order expansion of the scheme (67) gives

\[
\begin{align*}
\left( \tilde{u}^{x} \right)_{i,j}^{m+1} &= \left( \tilde{u}^{x} \right)_{i,j}^{m} + \left( \frac{\partial}{\partial y} \sum_{g=1}^{4} \left( 2\varphi^{(g)} - \Delta x r_g \partial_x \varphi^{(g)} - \Delta y l_g \partial_y \varphi^{(g)} \right) \Delta x \Delta y \right)_{i,j} \\
&+ O\left( h^3 \right),
\end{align*}
\]

where \( l_g \) and \( r_g \) are defined by the table

\[
\begin{array}{cccc}
l_g & g = 1 & g = 2 & g = 3 & g = 4 \\
r_g & 1 & 1 & -1 & -1 & 1
\end{array}
\]

and \( \varphi_x, \varphi_y \) are derivatives of \( \varphi \). Note that we changed the interpretation of \( \tilde{u}_{i,j} \) from cell mean value to point value in the middle of the cell. This switch introduces only an error of \( O(h^2) \) on both the left-hand and the right-hand sides of the equation. However, the leading expression within \( O(h^2) \) cancels on both sides and the remaining error is \( O(h^3) \).

By use of (22) with the definition of the flux function \( F \) in (41) we obtain the expansion of the exact solution

\[
u_{i,j}(t + \Delta t) = u_{i,j}(t) - \Delta t \left( \frac{\partial_y F}{-\partial_x F} \right)_{i,j} + \frac{\Delta t^2}{2} \left( \frac{\partial_y \left( v^{(x)} \partial_x F + v^{(y)} \partial_y F \right)}{-\partial_x \left( v^{(x)} \partial_x F + v^{(y)} \partial_y F \right)} \right)_{i,j} + O\left( \Delta t^3 \right).
\]

Since the Courant numbers are bounded we have \( O(h^3) = O(\Delta t^3) \). Hence, the direct comparison of exact and numerical increments of \( u_{i,j} \) yields the consistency condition

\[
\sum_{g=1}^{4} \left( 2\varphi^{(g)} - \Delta x r_g \partial_x \varphi^{(g)} - \Delta y l_g \partial_y \varphi^{(g)} \right) \Delta x \Delta y = -\Delta t F + \frac{\Delta t^2}{2} \left( v^{(x)} \partial_x F + v^{(y)} \partial_y F \right).
\]

This relation can be solved for \( \varphi^{(g)} \) by means of an ansatz with a first order and a second order contribution to \( \varphi^{(g)} \), viz.,

\[
\varphi^{(g)} = \frac{\Delta t}{\Delta x \Delta y} \varphi^{(g,1)} + \frac{\Delta t^2}{\Delta x \Delta y} \varphi^{(g,2)}.
\]

Introducing this into the consistency condition and comparison of coefficients of \( \Delta t \)-expressions leads us to

\[
\sum_{g=1}^{4} \varphi^{(g,1)} = -\frac{1}{2} F.
\]
which, of course, can be satisfied in many ways. We propose weights \( \omega^{(g)} \), yet unspecified, which sum up to unity and write

\[
\varphi^{(g,1)} = -\omega^{(g)} \frac{1}{2} F
\]

as first order flux distribution factor. By using this in our ansatz and further comparison of \( \Delta t^2 \)-coefficients in the consistency condition, the second order factor

\[
\varphi^{(g,2)} = \omega^{(g)} \frac{1}{4} \left( v(x) \partial_x F + v(y) \partial_y F - \sum_{g=1}^{4} \left( \frac{\Delta x}{\Delta t} r_g \partial_x (\omega^{(g)} F) + \frac{\Delta y}{\Delta t} l_g \partial_y (\omega^{(g)} F) \right) \right)
\]

is obtained.

In implementations we substitute the \( x \)- and \( y \)-derivatives in (70) by appropriate finite differences. Following the TVD analysis of one-dimensional methods, these finite differences need a limiting procedure in order to obtain nonoscillatory solutions. In the numerical experiments with discontinuous solutions we used the so-called WENO-limiter (see [17]), which is given by

\[
WENO(d_1, d_2) = \frac{d_1}{\sqrt{d_1^2 + \varepsilon}} + \frac{d_2}{\sqrt{d_2^2 + \varepsilon}}
\]

where \( d_1 \) and \( d_2 \) are left- and right-hand side finite differences and \( \varepsilon \) is a small number (\( \varepsilon \approx 10^{-8} \)). With use of this limiter we have

\[
\left. \frac{\partial \psi}{\partial x} \right|_{i} = \frac{WENO(\psi_i - \psi_{i-1}, \psi_{i+1} - \psi_i)}{\Delta x}
\]

for the limited derivative of a grid function \( \psi \).

### 4.3.2. Stability

The weights which have been introduced during the proof of consistency control the activation of the different basis flux distributions shown in Figure 1 (lower row). Their value should be chosen according to the direction of the advection. Clearly, one would not activate the first flux distribution \( \hat{\Phi}^{(1)} \) which is oriented towards the upper right if the wind is pointing in opposite direction. This would yield an unstable scheme. Indeed, stability is the issue that will specify the right choice of weights.

To investigate the stability we consider the one-sided scheme which uses only the first and the fourth flux distribution. Hence, it follows for the weights

\[
\omega^{(1)} \equiv \omega, \quad \omega^{(2,3)} = 0, \quad \omega^{(4)} = 1 - \omega
\]

with unknown \( \omega \). An impression of how different choices of \( \omega \) influences the stability of the scheme is given in Figure 2. It shows the contours of the maximal eigenvalue of the amplifier matrix \( \rho_{\text{max}} \) for different choices of \( \omega \) and different directions of the flow. The contour values and their shape have been obtained numerically for the first order scheme. We can see that a flow pointing exactly in the direction of a flux distribution \( (\theta_1 = 0 \text{ or } \theta_4 = 0) \) requires the activation of only the corresponding flux distribution \( (\omega_1 = 1 \text{ or } \omega_4 = 1) \) to yield stability. Furthermore \( \omega_1 = \omega_4 = 1/2 \) gives a stable scheme only for flows in the \( x \)-direction. This corresponds to the intuitive choices in these cases. In between these extreme cases Figure 2 indicates the existence of a single stable choice \( \bar{\omega}(\theta) \) for the weight.
The following lemma specifies this weight and the stability conditions of the one-sided scheme.

**Theorem 4.3 (stability).** Assume the advection velocity to be constant and $|v(x)| \neq 0$. Then, the one-sided, first order scheme consisting of flux distribution $\Phi^{(1)}$ and $\Phi^{(4)}$ with single weight $\omega$, time step $\Delta t$, and cells $\Delta x \times \Delta y$ is stable in the sense of a von Neumann analysis under the conditions

$$
\left( \frac{\Delta t v(y)}{\Delta y} \right)^2 \leq \frac{\Delta t v(x)}{\Delta x} \leq 1 \quad \text{and} \quad \omega \equiv \bar{\omega} = \frac{1}{2} \left( 1 + \frac{\Delta x v(y)}{\Delta y v(x)} \right).
$$

Under these stability conditions we have, furthermore, for the maximal spectral radius $\rho_{\text{max}}$ of the amplifier matrix

$$
\rho_{\text{max}}(\omega + \delta \omega) = 1 + c \delta \omega^2 + O(\delta \omega^3) \quad \text{with} \quad c > 0,
$$

i.e., the weight $\bar{\omega}$ is a local minimum of $\rho_{\text{max}}$.

**Proof.** We follow the stability analysis of von Neumann (see, e.g., [11]). The Fourier transform of the grid function $u_{i,j}$ is denoted by

$$
\hat{u}_{i,j} = \hat{u}_0 e^{i(\xi i + \eta j)}
$$

and introduced into the scheme (67) with (69) and (74), which leads to

$$
\hat{u}_{i,j}^{m+1} = \mathbf{T}_{\xi,\eta} \hat{u}_{i,j}^m.
$$

Here, $\mathbf{T}_{\xi,\eta}$ is the amplifier matrix of the scheme. The imaginary unit is denoted by $i = \sqrt{-1}$. Since the advection velocity is assumed to be constant the amplifier matrix has the form

$$
\mathbf{T}_{\xi,\eta} = \begin{pmatrix}
1 - b t^{(x)}(\xi, \eta, \omega) & a \alpha t^{(x)}(\xi, \eta, \omega) \\
-b \frac{1}{\alpha} t^{(y)}(\xi, \eta, \omega) & 1 + a t^{(y)}(\xi, \eta, \omega)
\end{pmatrix}
$$

with the Courant numbers $a = \frac{\Delta t v(x)}{\Delta x}$ and $b = \frac{\Delta t v(y)}{\Delta y}$ as well as $\alpha = \frac{\Delta x}{\Delta y}$. The functions $t^{(x)}$ and $t^{(y)}$ depend on $\xi$, $\eta$, and $\omega$ and follow from the scheme. For stability
the maximal spectral radius
\[ \rho_{\max}(a, b, \omega) = \max_{\xi, \eta \in (-\pi, \pi)} \rho(T_{\xi, \eta}) \]
has to be smaller or equal to unity. The geometry factor \( \alpha \) drops out during the calculation. Obviously, \( T_{\xi, \eta} \) has the eigenvector \((v(x), v(y))^T\) with eigenvalue \( \lambda_1 = 1 \), which corresponds to the first eigenvalue and eigenvector of the Jacobian given in (16) if the identity matrix is added. The second eigenvalue of \( T_{\xi, \eta} \) varies with \( \xi \) and \( \eta \). It has the form \( \lambda_2 = \tau_1(\eta) + \tau_2(\eta) e^{i\xi} \) with \( \tau_{1,2}(\eta) \in \mathbb{R} \), thus its maximal absolute value \( |\lambda_2| = |\tau_1(\eta)| + |\tau_2(\eta)| \) depends only on \( \eta \). A straightforward calculation leads to
\[
\rho_{\max}(a, \beta, \omega) = \max_{\eta \in (-\pi, \pi)} \left( \sqrt{(1 - a + a(1 + \beta - 2\omega \beta) \frac{1 - \cos \eta}{2})^2 + \frac{a^2}{4} (1 + \beta - 2\omega)^2 \sin^2 \eta} \right.
+ \sqrt{(a - a(1 - \beta + 2\omega \beta) \frac{1 - \cos \eta}{2})^2 + \frac{a^2}{4} (1 - \beta - 2\omega)^2 \sin^2 \eta} \),
\]
where we assumed \( |a| > 0 \) and defined the ratio \( \beta = \frac{b}{a} \).

The stable weight as defined in (75) can be written as
\[
\bar{\omega} = \frac{a + b}{2a} = \frac{1 + \beta}{2},
\]
which may be introduced into \( \rho_{\max} \), yielding
\[
\rho_{\max}(a, \beta, \bar{\omega}) = \max_{\eta \in (-\pi, \pi)} \left( |1 - a (1 - (1 - \beta^2) \frac{1 - \cos \eta}{2})| + |a| (1 - (1 - \beta^2) \frac{1 - \cos \eta}{2}) \right)
\]
after some calculation. The expression in large brackets is a positive quantity with the bounds
\[
0 \leq \min (1, \beta^2) \leq 1 - (1 - \beta^2) \frac{1 - \cos \eta}{2} \leq \max (1, \beta^2).
\]
Especially, for a given value of \( \beta \) there exists an \( \eta \) such that this expression is non-vanishing. From this fact we conclude for \( \rho_{\max}(a, \beta, \bar{\omega}) \)
\[
a < 0 \Rightarrow \rho_{\max} > 1;
\]
hence \( a \geq 0 \) is necessary for stability. The conditions for \( \rho_{\max} \leq 1 \) now follow from the condition that the modulus expression in \( \rho_{\max} \) should be nonnegative. Thus we obtain
\[
a (1 - (1 - \beta^2) \frac{1 - \cos \eta}{2}) \leq 1 \Rightarrow a \max (1, \beta^2) \leq 1,
\]
which, since \( \beta = b/a \), finally gives \( b^2 \leq a \leq 1 \) as stated in the lemma.

For the second part of the lemma we consider the Taylor expansion
\[
\rho_{\max}(a, \beta, \bar{\omega} + \delta \omega) = \rho_{\max}|_{\omega=\bar{\omega}} + \frac{\partial \rho_{\max}}{\partial \omega} \bigg|_{\omega=\bar{\omega}} \delta \omega + \frac{\partial^2 \rho_{\max}}{\partial \omega^2} \bigg|_{\omega=\bar{\omega}} \delta \omega^2 + \mathcal{O}(\delta \omega^3)
\]
for the maximal spectral radius. Under the conditions \( b^2 \leq a \leq 1 \) we have shown \( \rho_{\max}|_{\omega=\bar{\omega}} = 1 \). Starting with the general formula for \( \rho_{\max}(a, b, \omega) \) as given above, computer algebra software easily gives
\[
\frac{\partial \rho_{\max}}{\partial \omega}(a, \beta, \bar{\omega}) = 0
\]
and
\[ \frac{\partial^2 \rho_{\text{max}}(a, \beta, \bar{\omega})}{\partial \omega^2} = \frac{2a \sin^2 \eta}{1 + \beta^2 + (1 - \beta^2) \cos \eta} + \frac{2a^2 \sin^2 \eta}{|2 - a(1 + \beta^2) - (1 - \beta^2) \cos \eta|} \]
for the derivatives, which justifies the expansion with positive constant \( c \).

Note that stability of the one-sided scheme is given also for specific flows with \( \theta_1 < 0 \) or \( \theta_4 < 0 \) (according to Figure 2), which point outside the range given by the two flux distributions. Intuitively, we would expect \(|b| \leq a\) for the Courant numbers, but the lemma states only \(|b| \leq \sqrt{a}\). This condition becomes more and more restrictive if the angles \( \theta_{1,4} \) approach \(-\pi/4\). For the extreme cases \( \theta_{1,4} = -\pi/4 \) we would obtain a flow in a negative (respectively, positive) \( y \)-direction and the condition \(|b| \leq \sqrt{a} = 0\).

Furthermore, one of the weights becomes negative if \( \theta_1 < 0 \) or \( \theta_4 < 0 \) holds.

Lemma 4.1 investigates the first order scheme. The analysis for the second order scheme becomes much more involved and hardly solvable by hand. But numerical experiments suggest that the second order scheme appears to remain stable under the same conditions. Furthermore the numerical exploration of the amplifier matrix results in a picture very similar to the right-hand side of Figure 2. A detailed investigation of the second order scheme remains for future work.

Finally, we generalize the result for the one-sided scheme to the full upwind scheme with four flux distributions. The one-sided scheme may easily be formulated for all four possible coordinate directions. For a general scheme, we propose a superposition of these four one-sided schemes in order to obtain a full upwind scheme. Hence, for any flow the weights are chosen such that two flux distributions are activated according to the appropriate one-sided case. The resulting weights may be constructed from the direction vector each skeleton is associated with. These vectors are given by

\[ n_1 = (1, 1), \quad n_2 = (-1, 1), \quad n_3 = (-1, -1), \quad n_4 = (1, -1), \]

following the numbering of the sketch in Figure 1. Based on these vectors the general local weights \( \omega^{(1,2,3,4)} \) have the representation

\[ \omega^{(g)}(v_{i,j}) = \frac{\max(n_g \cdot v_{i,j}, 0)}{\sum_{\gamma=1}^{4} \max(n_\gamma \cdot v_{i,j}, 0)}, \]

which may be verified to coincide with the appropriate one-sided case depending on the direction of \( v \). In addition we define \( \omega^{(g)}(0) = 0 \). By extrapolation of Lemma 4 we may draw the conclusion that the scheme (67) with the weights (78) will be stable provided we have

\[ \max_{x,y \in \Omega} (|a_{i,j}|) \leq 1 \quad \text{and} \quad \max_{x,y \in \Omega} (|b_{i,j}|) \leq 1, \]

where \( a_{i,j} \) and \( b_{i,j} \) are local Courant numbers. One of the weights of (78) is displayed in Figure 3 as the dark curve. Note the correspondence of the shapes between the curve in this figure and the contour in Figure 2. Unfortunately, the weight given in (78) is not differentiable at points where \( v \) is orthogonal to any of the \( n_g \) due to the function \( \max(\cdot, 0) \). However, at least one continuous derivative is needed for second order accuracy as stated in the remark following Lemma 3.2. As regularization of \( \max(\cdot, 0) \), we propose

\[ \max_\varepsilon(x, 0) = \frac{1}{2}(x + \sqrt{x^2 + 4\varepsilon}) \]
for use in (78) resulting in a regularized weight $\bar{\omega}_\varepsilon$. The curve of $\bar{\omega}_\varepsilon$ is also shown in Figure 3. Note that this weight gives a deviation from the weight $\bar{\omega}$ obtained in Lemma 4.1. However, if we choose $\varepsilon = h = \max(\Delta x, \Delta y)$ we have

$$\rho_{\text{max}}(\bar{\omega}_\varepsilon) \approx \rho_{\text{max}}(\bar{\omega} + \sqrt{h}) \approx 1 + c h$$

according to the second statement in Lemma 4.1. This increases the error constant of the scheme but still gives stability; see, e.g., [11].

Another possible regularization is given by

$$\tilde{\omega}^{(g)}(v) = \max(n_g \cdot v, 0)^2 \left/ \frac{1}{2} \right. \|v\|^2,$$

which is also depicted in Figure 3. This weight deviates considerably from $\bar{\omega}$ and stability is not assured by Lemma 4.1. However, we want to remark that in our numerical calculations with this weight instabilities did not occur. This fact needs further investigation. It could be possible that error modes considered by the von Neumann analysis are not excited in the numerical evaluations due to the constraint-preserving property.

5. Numerical experiments. We proceed to present numerical experiments for two-dimensional div-preserving advection (22) calculated with the upwind scheme given in (67). This scheme exactly preserves the extended divergence operator $\tilde{C}(\cdot)$. The symmetric schemes (48), which preserves the classical operator and (58) are not considered due to their instability. For the scheme (67) we write $FD$, which is an abbreviation of “flux distribution.” $FD^{(2)}$ stands for the second order scheme (70) with weight $\bar{\omega}$, while $FD^{(2)}$ uses the regularized weight $\bar{\omega}_\varepsilon$. Analogously, $FD^{(1)}$ denotes the first order scheme (69) with weight $\bar{\omega}$. For the smooth test cases we used central finite differences to approximate the derivatives in the second order flux distribution coefficient given by (70).

5.1. Smooth test cases. In order to obtain empirical orders of convergence we considered smooth initial conditions

$$u_0(x, y) = \begin{pmatrix} -1 + \frac{1}{2} \sin(\pi x) + \frac{1}{4} \cos(\pi y) \\ 1 + \frac{1}{2} \cos(\pi x) + \frac{1}{4} \sin(\pi y) \end{pmatrix}$$
in the computational domain $\Omega = [-1, 1]^2$. To eliminate the influence of boundary conditions periodic boundaries were furnished in both dimensions. The initial vector field $u_0$ has a nonvanishing divergence which will be frozen under div-preserving advection. The vector field is advected by the velocity field

$$v(x, y) = \left(1 + \frac{1}{4} \cos(\pi x) + \frac{1}{2} \sin(\pi y) \right) \left(1 + \frac{1}{4} \sin(\pi x) + \frac{1}{2} \cos(\pi y) \right),$$

which is periodic as $u_0$. As end time, $t = 0.5$ was chosen. Since an analytic solution is not available for this case, a reference solution has been calculated on a uniform grid with $1200 \times 1200$ points and 540 constant time steps. The maximal Courant number

$$c_{\text{max}} = \max_{x,y \in \Omega} \left( \frac{v(x) \Delta t}{\Delta x}, \frac{v(y) \Delta t}{\Delta y} \right)$$

for this solution was approximately 0.97.

The reference solution is used to calculate empirical orders of convergence for calculations on $N \times N$ grids with $N = 10, 20, 30, 40, 50, 60, 80, 100, 120, 150, 200$. All these calculations were performed with constant time steps such that $c_{\text{max}} \approx 0.875$. For the coarsest grid with $10 \times 10$ this results in five time steps. The left-hand side of Figure 4 shows the $L^1$-errors of second order schemes with regularized and non-regularized weight as well as the $L^1$-errors of the first order scheme. As predicted in the preceding section, the $FD^{(2)}$-scheme does not achieve full second order. Only due to the regularization (80) full second order is obtained with the $FD^{(2)}_{\varepsilon}$-scheme. The errors and the order of convergence depend slightly on the regularization parameter $\varepsilon$. In Figure 4, $\varepsilon = 5 \Delta x$ was chosen. Higher values give a slightly improved order of convergence. $FD^{(1)}$ exhibits first order independently of the regularity of the weight.

The right-hand side of Figure 4 displays the $L^\infty$-error of discrete divergences of the $FD^{(2)}_{\varepsilon}$-solution at $t = 0.5$. The curves refer to evaluations with the classical and the extended operator, $\tilde{C}^{(0)}$ and $\tilde{C}^{(\star)}$ as given in (45) and (53), respectively. Due to the constraint preservation of all $FD$-schemes, the evaluation of the extended operator $\tilde{C}^{(\star)}$ yields the same numerical value for the divergence during the entire calculation. This value is given by the discrete initial conditions. Hence, the lower curve in Figure 4 (right) simply represents the increasing resolution of the initial
conditions and demonstrates second order for the extended operator. In contrast, the value of the classical operator is affected during the numerical calculation (not shown). However, since the solution is smooth the evaluation of \( \tilde{C}(0) \) at \( t = 0.5 \) is converging, which is visible in the right plot (upper curve) of Figure 4.

5.1.1. Box test case. If initial conditions for the div-preserving advection (22) are given in the form

\[
\mathbf{u}_0(x, y) = (0, g'(x) g'(y))
\]

with derivatives of a function \( g \) and the velocity field by \( \mathbf{v}(x, y) = (1, 0) \), i.e., pointing constantly in the \( x \)-direction, the exact solution has the form

\[
\mathbf{u}(x, y, t) = (g''(y)(g(x) - g(x - t)), g'(x - t) g'(y)).
\]

As an example we choose

\[
g(x) = \begin{cases} 
\frac{16}{5} s^5 - \frac{8}{3} s^3 + s, & \text{if } -\frac{1}{2} < s < \frac{1}{2}, \\
\pm \frac{4}{15} & \text{else,}
\end{cases}
\]
so that the initial field \( u_0 \) is nonvanishing only inside the box \((-1/2, 1/2)^2\). The initial condition has a nonvanishing divergence; hence, the advection will differ from ordinary advection, though the advection velocity is constant.

In the numerical test the system given by these initial conditions is rotated by 45° such that advection takes place in a diagonal grid direction. The initial conditions are displayed in the upper left corner of Figure 5. The center of the box is moved to \((-0.1, -0.1)\). The contour shading represents values of \( ||\tilde{u}|| \), which ranges form zero to 1.36, while the lines in the plots represent field lines of \( \tilde{u} \) (the flow which is induced by \( \tilde{u} \)). The calculation is conducted in the domain \([-1, 1]^2\) with constant extrapolation in the boundary cells. Besides the initial conditions, Figure 5 displays numerical results at \( t = 0.5 \) for three uniform meshes with different resolutions. All results were obtained with the \( FD^{(2)} \) scheme with constant time steps such that the maximal courant number \( c_{\text{max}} \approx 0.884 \). Note that the weight is constant in this example since the advection velocity is constant. In the exact solutions the field lines are bent outside the box due to the advection. Since the divergence of \( \tilde{u} \) does not vanish initially, the field lines fill up the way and their starting and ending points stay inside the initial box. In other words, the nonvanishing divergence inside the box acts as source and sink for the field lines.

On the coarse grid the solution is spoiled at the sides by artificial field lines. These field lines correspond to values of \( \tilde{u} \) in the magnitude of the truncation error which are introduced by the finite stencil of the scheme at the boundary of the box. Note that outside the initial box erroneous field lines appear as closed lines which indicates the solenoidal character of the scheme. On the fine mesh the solution is well resolved.

In Figure 6 we display the \( L^1 \)-error of the variable \( u \) and the \( L^\infty \)-error of the divergence for the box test case together with averaged empirical orders of convergence. Second order is well obtained, while the \( FD^{(1)} \)-scheme shows a slight superconvergence for this example. The irregularities in the second order error curve might be due to the nonsmooth gradient of the solution (87) with (88) along the lines \( y = \pm\frac{1}{2} \). This is also the reason that the convergence of the divergence on the right-hand side of the figure is reduced to first order. Like in the smooth test case, the divergence error for the extended operator gives the same value during the entire simulation since this value is locally preserved by the scheme. The method freezes the discrete divergence of the initial conditions like the analytical system does.
Box test case for curl-preserving advection. It is interesting to ask for the dual solution of the box test case in the sense of the duality of curl-preserving and div-preserving advection as indicated in (25). The solution is depicted in Figure 7, which should be directly compared with Figure 5. The dual solution is obtained by taking the orthogonal complement of the initial conditions (86) as well as of the solution (87). The result is a solution of the curl-preserving advection (23). Accordingly, the field lines in Figures 5 and 7 are orthogonal.

The plots in Figure 7 can be obtained equivalently in two ways: either by taking the orthogonal vector in each cell of the result of the scheme for div-preserving advection or by constructing the corresponding flux distribution scheme for curl-preserving advection and applying it to the dual initial conditions. In fact, this scheme would differ from the div-preserving scheme only in the structure of the flux distribution shape functions in (55). These shape functions are substituted by their orthogonal complement, yielding outward pointing arrows instead of the approximate loops in Figure 1. The resulting scheme preserves perfectly the discrete value of the curl but has the same properties in consistency and stability as its dual counterpart, which was constructed in the preceding sections. Indeed the plot of errors and the empirical
5.1.2. Rotating hump. The advection velocity $\mathbf{v}(x, y) = (-y, x)^T$ results in a rotational flow around the origin. As mentioned in section 2.4, the components of $\mathbf{u}$ are not ordinarily advected in such a flow if div-preserving advection is considered. Indeed, the exact solution for divergence-preserving advection given in (22) with a rotational flow is given by

$$u(x, t) = R(t)^{-1} u_0(R(t)x),$$

where $R(t)$ is a orthogonal matrix which rotates a vector by the angle $t$ and $u_0$ is the initial condition. In the case of ordinary advection the inverse $R^{-1}$ would be missing in the solution. However, if $\mathbf{u}$ has initially vanishing divergence, the 2-norm $||\mathbf{u}||$ satisfies an ordinary advection equation, which follows directly from (89).

We consider the initial condition

$$u_0(x, y) = \frac{1}{5\varepsilon} \begin{pmatrix} -y \\ x - \frac{1}{2} \end{pmatrix} \exp \left( -\frac{(x - \frac{1}{2})^2 + y^2}{\varepsilon} \right)$$

with $\varepsilon = \frac{1}{20}$. This vector field is easily verified to be solenoidal. It produces field lines circling around $(\frac{1}{2}, 0)$ and a smooth but distinct hump in $||\mathbf{u}||$ with an essential radius of approximately $\frac{\varepsilon}{2}$. Note that in the center of this hump, $||\mathbf{u}||$ is zero. The computations are conducted in the domain $[-1, 1]^2$, where the exact solution is prescribed in the ghost cells of the boundary.

Two numerical solutions of the problem calculated with $FD_x^{(2)}$ are depicted in Figure 8 at time $t = \pi$. The Courant number in these calculations was 0.963. The figure shows contours and contour lines of the absolute value $||\tilde{u}||$ for results obtained with two different grids, $20 \times 20$ and $80 \times 80$ cells. It also displays the loss of height of the hump compared to the exact solution. The fine grid calculation exhibits a good preservation of symmetry and height.

5.2. Calculating discontinuities. Finally, we present numerical experiments with discontinuous solutions. Discontinuities are most challenging for divergence-
preserving methods in the context of magnetohydrodynamics where the magnetic field jumps across shock waves; see, e.g., [6], [26].

5.2.1. Horizontal and diagonal direction. We consider constant advection in the $x$-direction, i.e., $\mathbf{v}(x,y) = \left( \frac{3}{4}, 0 \right)^T$. The initial vector field is given by

$$\mathbf{u}_0(x,y) = \begin{pmatrix} 1.0 \\ 0.3 + 1.2 h(x) \end{pmatrix}$$

with the Heaviside function $h(x)$, which is zero for $x \leq 0$ and unity if $x$ is positive. In both half spaces $x \leq 0$ the vector field is smooth and its divergence is zero. Furthermore, across the discontinuity the normal component of $\mathbf{u}_0$ remains constant, which leads to zero divergence in the weak sense. The vector field $\mathbf{u}_0$ mimics the behavior of the magnetic field in a magnetohydrodynamic shock wave. As the divergence vanishes and the advection is constant, the discontinuity will be linearly advected. The setting will be varied by rotation with an angle $\theta$. Horizontal advection corresponds to $\theta = 0$.

The problem at hand is calculated in $\Omega = [-1,1]^2$ with the second order scheme $FD^{(2)}$ on a grid with $100 \times 100$ cells up to time $t = 0.9$. Ghost cells at the boundary are filled by constant extrapolation and adjustment according to the angle $\theta$. In Figure 9 we display the results for horizontal ($\theta = 0$) and diagonal ($\theta = 45^\circ$) advection. Both problems have been computed either with central finite difference or limited differences (73). The time step for both horizontal and diagonal advection was chosen after a Courant number of approximately 0.96; hence the horizontal advection took more time steps due to a more restrictive stability condition. The figure shows the absolute value $\|\tilde{\mathbf{u}}\|$ by following cuts of the solutions normal to the discontinuities. The solutions with central finite differences exhibit familiar oscillations which are eliminated by the use of the limiter. The discontinuities are well resolved. Note the slight asymmetry in the profiles of the discontinuities compared to the exact solution,
Fig. 10. Divergence-cleaned and uncleaned solutions of the discontinuous test example in the case of a rotation with $\theta = 26.6^\circ$ (upper row). Below, the discrete initial divergence of both cases are displayed. The strong deviation from zero leads to a misfit of the computed solution. A divergence cleaning procedure applied to the initial conditions removes the disagreement. Both computations are conducted with use of WENO limitation.

which is drawn as a thin line in Figure 9. This is due to displaying $|\tilde{u}|$ instead of the components $\tilde{u}(x)$ or $\tilde{u}(y)$.

5.2.2. Oblique directions and initial divergence cleaning. The evaluations of both the extended and the classical divergence operator give exactly zero for the initial conditions of the horizontal or diagonal discontinuity. This comes due to symmetry. For discontinuities in all other noncoordinate and nondiagonal directions this is no longer true. Though the analytic initial condition is divergence-free, the discrete evaluation of the divergence in the vicinity of the discontinuity leads to significant deviations from zero.

The left-hand side of Figure 10 shows the result of the computation with $\theta \approx 26.6^\circ$ which corresponds to $\tan \theta = \frac{1}{2}$. The upper right plot displays the run of $|\tilde{u}|$ along a normal cut and exhibits a complete disagreement with the exact solution (thin line). The plot below shows the evaluation of the extended divergence operator $\tilde{C}(\tilde{u})$ along the same cut of the initial conditions. Due to the constraint preservation this curve stays the same for all time steps. The strong deviations of the divergence from zero are responsible for the disagreement of the computed with the exact solution. Hence, the upper right plot does not represent a failure of the method, but rather indicates the high quality of the constraint preservation. In fact, the computed result belongs to a solution for analytic initial conditions whose divergence is disturbed according to the curve in the lower right plot.

In order to get rid of the divergence in the initial conditions, the discrete field has to be corrected as proposed, e.g., in [2]. We stress that this cleaning procedure is only needed for initial conditions with nonvanishing divergence due to discontinuities. Hence, the procedure is only applied once in the beginning of the calculation. If the discrete initial divergence is zero, it stays zero due to the properties of our scheme.
The cleaning procedure solves the elliptic equations

\[
\text{div grad } \psi = \text{div } \tilde{u} \quad \text{in } \Omega,
\]
\[
\psi = 0 \quad \text{on } \partial \Omega
\]

for the auxiliary discrete field \( \psi \). The discrete initial field \( \tilde{u} \) is afterwards corrected by \( \tilde{u} \rightarrow \tilde{u} - \text{grad } \psi \) which gives a discrete solenoidal field. This procedure represents the projection onto the divergence-free space (Hodge projection). The differential operator \( \text{div grad} \equiv \triangle \) has to be built from the extended divergence operator (53) since the result should give a divergence-free field according the extended operator. The use of the traditional discretization of Laplace operator will not lead to this property. The construction of the Laplace operator by applying an appropriate discrete gradient and afterwards \( \tilde{C}(\ast) \) to the field \( \psi \) results in a special discrete Laplace operator which assures that the evaluation of \( \tilde{C}(\ast) \) on the corrected solution will be zero. The discretized form of (92) may be solved by using iterative linear solvers.

The discrete divergence of the corrected initial condition (91) in the case of \( \theta = 26.6^\circ \) is shown in the lower right plot of Figure 10. Note the scale of the ordinate. Finally, the approximation ability of the scheme is fully revealed as can be seen in the upper right plot of Figure 10. The small fluctuations visible in the solution are introduced by the initial cleaning procedure, they vanish with grid refinement.

Note that during the cleaning procedure based on the extended operator \( \tilde{C}(\ast) \) we have no control over the value of the classical operator (45). Correspondingly, the value of the divergence obtained with this operator is not vanishing if evaluated for the initial conditions. It will also vary during the time steps of the flux distribution scheme. However, the maximal value remains finite during the calculation independent of the grid size as is visible in Figure 11. Moreover, the value of the classical operator decreases due to the numerical smoothing of the discontinuous solution.

6. Sketch of the method in 3 dimensions. We will shortly give a sketch how to extend the constraint-preserving method to the three-dimensional case. The presentation will not be exhaustive but will provide evidence that three-dimensional methods may be constructed from the presented framework as well.

We restrict ourself to div-preserving advection, given in (4)_{curl}. In three dimensions, methods for curl-preserving advection cannot be obtained by duality but need extra considerations. Furthermore, the most important application of curl-preserving advection is the shallow water system which is restricted to two dimensions.
6.1. Flux distributions. For discrete divergence operators in three dimensions a representation similar to that of Lemma 2 may be found. However, in this case there exists a family of operators with 17 parameters which is quite involved. Inspired by the two-dimensional results, we proceed by generalizing the extended operator (53) directly to three dimensions, obtaining

\[
\begin{align*}
\tilde{C}_K^{(\ast)} \tilde{u} &= \left\{ \tilde{u}_{i+1,j,k}^{(x)} \right\}_{g,z} - \left\{ \tilde{u}_{i-1,j,k}^{(x)} \right\}_{g,z} \\
&\quad + \frac{\left\{ \tilde{u}_{i,j+1,k}^{(y)} \right\}_{x,z} - \left\{ \tilde{u}_{i,j-1,k}^{(y)} \right\}_{x,z}}{2\Delta y} \\
&\quad + \frac{\left\{ \tilde{u}_{i,j,k+1}^{(z)} \right\}_{x,y} - \left\{ \tilde{u}_{i,j,k-1}^{(z)} \right\}_{x,y}}{2\Delta z},
\end{align*}
\]

(93)

where curled brackets this time stand for

\[
\begin{align*}
\left\{ \psi_{i,j,k} \right\}_{g,z} &= \frac{1}{16} (4\psi_{i,j,k} + 2\psi_{i,j+1,k} + 2\psi_{i,j-1,k} + 2\psi_{i,j,k+1} + 2\psi_{i,j+1,k-1} + 2\psi_{i,j-1,k-1} + 2\psi_{i,j,k+1} + 2\psi_{i,j+1,k-1} + 2\psi_{i,j-1,k+1} + 2\psi_{i,j,k-1} + 2\psi_{i,j+1,k+1} + 2\psi_{i,j-1,k+1} + 2\psi_{i,j,k-1} + 2\psi_{i,j+1,k+1} + 2\psi_{i,j-1,k-1} + 2\psi_{i,j,k+1} + 2\psi_{i,j+1,k-1} + 2\psi_{i,j-1,k+1} + 2\psi_{i,j,k-1} + 2\psi_{i,j+1,k+1} + 2\psi_{i,j-1,k+1} + 2\psi_{i,j,k-1} + 2\psi_{i,j+1,k+1} + 2\psi_{i,j-1,k-1} + 2\psi_{i,j,k+1}),
\end{align*}
\]

(94)

i.e., plane-wise averaging. Solving the linear system (39) now gives possible shape functions for flux distributions. All the resulting skeletons have essentially the two-dimensional shape given in (55) and depicted in Figure 1, except they now come with three possible orientations, approximating a circle either in the \((x, y)\)-plane, the \((x, z)\)-plane, or the \((y, z)\)-plane. Hence, there are 36 possible flux distributions altogether, four circles in each cut of the \(3 \times 3 \times 3\) grid box. Three of them are sketched on the left-hand side of Figure 12.

Note that it is necessary to take at least three flux distributions to construct a three-dimensional method, since the flux \(F = u \times v\) in (4)\(_{\text{curl}}\) now has three independent components.

6.2. Possible methods. The dimensionally split character of the three-dimensional flux distributions leads to a method which uses directly the two-dimensional
results. Indeed, the evolution equation (4) can be split into three two-dimensional operators as well. We write

\[
\frac{\partial}{\partial t} \begin{pmatrix} u^{(x)} \\ u^{(y)} \\ u^{(z)} \end{pmatrix} + \begin{pmatrix} \partial_x F^{(x)} \\ \partial_y F^{(y)} \\ \partial_z F^{(z)} \end{pmatrix} = 0,
\]

where \( \mathbf{F} = (F^{(x)}, F^{(y)}, F^{(z)})^T = \mathbf{u} \times \mathbf{v} \) represents the flux function. It becomes obvious that each bracket can be discretized by the two-dimensional method (67). The resulting flux across a corner is represented by the left picture in Figure 12. The procedure is similar to the operator splitting approach where each flux in a multidimensional conservation law is discretized in a one-dimensional manner (see e.g., [11]), except here we use two-dimensional methods for the single operators. Nevertheless, we expect loss of stability since the cell directly across the corner (see Figure 12, left) is not affected in a single time step. Possible and straightforward help would be to use a fractional time step method, e.g., with Strang splitting, which updates the brackets in (95) successively.

To circumvent the use of splitting it is possible to construct a fully three-dimensional flux distribution as sketched in Figure 12 (right). These flux distributions result from averaging each flux distribution on the left-hand side of the figure with its counterpart in the neighboring parallel grid plane (not shown). A scheme using this single set of flux distributions has the form

\[
\begin{pmatrix} \hat{u}^{(x)} \\ \hat{u}^{(y)} \\ \hat{u}^{(z)} \end{pmatrix}_{i,j,k}^{m+1} = \begin{pmatrix} \hat{u}^{(x)} \\ \hat{u}^{(y)} \\ \hat{u}^{(z)} \end{pmatrix}_{i,j,k}^m + \begin{pmatrix} \delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(x,1)}) \Delta x - \delta^{(z)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(y,1)}) \Delta x & -\delta^{(z)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(y,1)}) \Delta y + \delta^{(z)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(x,1)}) \Delta z & \delta^{(z)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(y,1)}) \Delta x - \delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(x,1)}) \Delta y + \delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi^{(x,1)}) \Delta z \end{pmatrix} \varphi_{i,j,k},
\]

where we used the abbreviations

\[
\begin{align*}
\delta^{(x)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi) &= \varphi_{i+1,j+1,k} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k}, \\
\delta^{(y)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi) &= \varphi_{i+1,j+1,k} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k}, \\
\delta^{(z)}_{i+\frac{1}{2},j+\frac{1}{2},k} (\varphi) &= \varphi_{i+1,j+1,k} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k+1} + \varphi_{i+1,j+1,k-1} + \varphi_{i+1,j+1,k}.
\end{align*}
\]

Analogously one has to incorporate the flux distributions for the rest of the corners of the grid cell. First order consistency for the single set of flux distributions in (96) leads to

\[
\begin{pmatrix} \varphi^{(x,1)} \\ \varphi^{(y,1)} \\ \varphi^{(z,1)} \end{pmatrix} = -\frac{\Delta t}{4\Delta x \Delta y \Delta z} \begin{pmatrix} \Delta x F^{(x)} \\ \Delta y F^{(y)} \\ \Delta z F^{(z)} \end{pmatrix},
\]

which gives a method for div-free advection in three dimensions which exactly preserves the discrete value of the divergence evaluated by (93). The flux distributions
of the rest of the corners could be incorporated by weights as in the two-dimensional case. The elaboration of the details of the three-dimensional method remains for future work.

7. Conclusions. In this paper we drew attention to constraint-preserving advection equations. These equations are characterized by the existence of an intrinsic differential constraint which holds locally during the evolution. They form models for general evolution equations with constraints which can be found in various fields of physics and engineering.

Starting from the hypothesis that numerical methods should respect the constraints, we proposed a general framework for constructing constraint-preserving schemes. Based on this framework a multidimensional upwind method was developed. Consistency and stability were proven, and various numerical experiments demonstrated the ability and reliability of the new scheme. We also re-derived former numerical schemes within our framework. The new method relies on special flux distribution and does not require staggered grids, time-step-wise global correction procedures, or modified evolution equations as proposed in former approaches, e.g., [8], [2], [6].

In [18] a precursor of the present method was used in the context of the method of transport [9], [10] to solve the magnetic evolution part of a magnetohydrodynamic computation. In [25] the results of this paper are used to derive general divergence-preserving finite-volume schemes for magnetohydrodynamics. Future work will also include applications to electrodynamics, meteorological flows, and Einstein equations.

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In this work we considered a rectangular mesh as a first approach. The treatment of more general grids, e.g., triangular or quadrilateral, is a major issue for future work. The framework given in this paper allows for constraint-preserving methods on such grids. The main problem is to find an appropriate discretization of the constraint on the given grid. In [25] divergence-preserving methods on triangular grids are derived using the framework of this paper. In [7] an approach to triangular grids is presented based on staggered grids.

The discrete constraint preservations also requires further investigations on discrete data treatment. Implementations of boundary conditions as well as restriction and prolongation in an adaptive grid (see [27]) should be revised from the angle of constraint preservation.

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REFERENCES


