Slow gas microflow past a sphere: Analytical solution based on moment equations

Manuel Torrilhon

Seminar for Applied Mathematics, ETH Zurich, Zurich 8092, Switzerland

(Received 17 February 2010; accepted 6 May 2010; published online 15 July 2010)

The regularized 13-moment equations are solved analytically for the microflow of a gas past a sphere in the case of low Mach numbers. The result is given in fully explicit expressions and shows nontrivial behavior for all fluid fields including stress, heat flux, and temperature. Various aspects of the flow such as temperature polarization and total force are reproduced correctly for moderate Knudsen number. The analytical solution allows studying the rise of Knudsen layers and their interaction and coupling to the fluid variables in the bulk. Additionally, based on the regularized 13-moment equations system, hybrid boundary conditions are given for the standard Stokes equations in order to enable them to predict nonequilibrium effects in the flow past a sphere. © 2010 American Institute of Physics. [doi:10.1063/1.3453707]

I. INTRODUCTION

The regularized 13-moment (R13) equations are using moment approximations in kinetic gas theory as introduced by Grad to describe gas flows when the Knudsen number, the ratio between mean free path and observation scale, becomes significant. Classical fluid theories such as the constitutive laws of Navier–Stokes and Fourier are valid only close to equilibrium and fail for processes at Knudsen numbers as low as 0.01 because there are not sufficient particle collisions within the gas to maintain equilibrium. Hence, it is well known that for larger Knudsen numbers, classical gas dynamics cannot be used to describe flows in microsettings or rarefied situations.

The derivation of the R13 equations relies on the combination of moment approximations and asymptotic expansions in kinetic gas theory. The system has been shown to be stable and of high asymptotic accuracy yet using only a relatively small set of variables, namely, density, velocity, temperature, stress deviator, and heat flux. It has been demonstrated to succeed on the prediction of various nonequilibrium processes, for example, shock waves or channel flows, exhibits desirable mathematical features like an entropy, and is easy to use in numerical simulations. An overview of the equations and their features can be found in Ref. 11, the special issues, and in the text book. The R13 equations deliver quantitatively correct results for Knudsen numbers up to 0.5 or, in some processes, up to Kn ≈ 1. For Kn ≈ 1, at least qualitative features can be expected to be correctly reproduced. Variants of the regularized equations have been considered in Ref. 14 with less moments and in Refs. 15 and 16 with more moments. More moments give a more accurate description of the flow at the expense of an increasing complexity of the equations.

A distinguished advantage of the R13 system is the possibility to formulate valid boundary conditions derived from kinetic gas theory. Originally proposed in Ref. 1 for classical moment equations, pioneered in Ref. 17 for the R13 equations, and refined in Ref. 18, the boundary conditions allow the new set of equations to be applied to many gas dynamic test cases and processes.

This paper considers the high-Knudsen-number slow flow of a gas past a sphere and solves the linearized R13 equations analytically for this case. Known as the Stokes flow around a sphere, this process is well known as a standard text book example in fluid dynamics, see, e.g., Refs. 19 and 20. However, the flow of a rarefied gas, or in microsituations where the Knudsen number based on the mean free path of the inflow and the radius of the sphere is large, will differ from the standard solution. One interesting effect is the temperature polarization as shown in Refs. 21 and 22. Without nonlinear dissipation and external temperature differences, the flow in general produces a temperature pattern which shows a different temperature in front and in the back of the sphere. Knudsen layers are present and lead to unexpected patterns in the heat flow away from the sphere and on the surface.

The rarefied or microsituation for a flow past a sphere has been considered for the 13-moment-case in a Ph.D. thesis during the early beginnings of moment equations. The paper solves the linearized Boltzmann equation numerically for large and very large values of the Knudsen numbers and shows various features of the flow. Slip-flow past a sphere based on Navier–Stokes equations has been considered, for example, in Refs. 24 and 25. In general, such an approach is unable to describe temperature polarization and also predicts a wrong total force for large Knudsen numbers. High speed rarefied flows past obstacles have been studied in, for example, Ref. 26.

The results of this paper demonstrate the capabilities of the R13 system to describe full-scale microflows beyond simplified one-dimensional situations. The solution contains a complete set of Knudsen layers, jumps and slips at the boundary, as well as typical bulk effects from nongradient...
transport phenomena. Though limited to moderate Knudsen numbers, the essential qualitative effects observable in the flow past a sphere are correctly reproduced. Furthermore, the availability of explicit analytical expressions provides physical insight into the intriguing flow patterns and helps to understand rarefaction effects. This particular aspect of a continuum model like the R13 equations goes beyond any kinetic simulation result. In a next step, the enlarged R26 moment system of Ref. 15 could be used and solved analytically in the same way. The result would show an even increased accuracy in terms of Knudsen number.

In the following, we will introduce the linearized, steady R13 equations and appropriate boundary conditions. Section III specializes the equation for the geometry of a sphere and gives a reduced set of ordinary differential equations to be solved as boundary value problem. This section also discusses the classical first order solution of Stokes for comparison. Explicit results and a discussion of the flow field of various variables are given in Sec. IV. Section V gives enhanced hybrid boundary conditions for the Stokes equations and presents detailed comparisons between the hybrid solution and R13. The paper ends with a conclusion and an appendix listing special differential operators in spherical geometry.

II. LINEAR, STEADY R13 EQUATIONS

The fully nonlinear R13 equations have been derived in Ref. 5 and well-posed boundary conditions are developed in Ref. 18. Here we are interested in the linear steady case.

A. Equations

The R13 equations describe the gas using the variables density \( \rho \), velocity \( \mathbf{v} \), temperature \( \theta \) (in energy units \( \theta = RT \)), stress deviator \( \mathbf{\sigma} \), and heat flux \( \mathbf{q} \). The equation of state gives the pressure \( p = \rho \theta \). We consider a ground state given by \( \rho_0 \), \( \theta_0 \), and \( p_0 \), in which the gas is at rest and in equilibrium with vanishing stress deviator and heat flux. The linearized equations describe deviations from this ground state and from now on we use the variables \( (\rho, \mathbf{v}, \theta, \mathbf{\sigma}, \mathbf{q}) \) for these deviations.

The linearized conservation laws of mass, momentum and energy are given by

\[
\nabla \cdot \mathbf{v} = 0, \quad (1a)
\]

\[
\nabla \rho + \nabla \cdot \mathbf{\sigma} = 0, \quad (1b)
\]

\[
\nabla \cdot \mathbf{q} = 0, \quad (1c)
\]

where the stress deviator and heat flux need to be specified. For these, the R13 system gives the constitutive relations

\[
\nabla (\mathbf{\sigma})_{\text{sym}} + 2 p_0 (\nabla \mathbf{v})_{\text{sym}} = -\frac{\rho_0}{\mu_0} \mathbf{\sigma} + \frac{2 \mu_0}{3 \rho_0} \nabla \cdot (\nabla \cdot \mathbf{\sigma}) = \nabla \cdot \nabla (\mathbf{\sigma})_{\text{sym}} - \frac{1}{3} \nabla \cdot ((\nabla \cdot \mathbf{\sigma}) I), \quad (2)
\]

\[
\theta_0 \nabla \cdot \mathbf{\sigma} + \frac{5}{2} p_0 \nabla \theta = -\frac{2 \rho_0}{3 \mu_0} \mathbf{q} + \frac{6 \mu_0}{5 p_0} \Delta \mathbf{q}, \quad (3)
\]

where the parameter \( \mu_0 \) is the viscosity of the gas at ground state. These equations involve differential operators for stress and heat flux. Hence, the conservation laws (1) must be solved together with Eqs. (2) and (3) as a coupled system of partial differential equations. In Eq. (2), we use the notation \( (A)_{\text{sym}} = \frac{1}{2}(A + A^T) \) for matrices \( A \). Note that \( \nabla (\nabla \cdot \mathbf{\sigma}) \) is the gradient of the divergence of the stress deviator and, as such, a matrix. We view the conservation laws as equations for pressure, velocity, and temperature. The density can be computed from the solution by means of the equation of state.

The viscosity gives rise to the mean free path \( \lambda_0 \) in the gas at ground state from which we define the Knudsen number by

\[
\text{Kn} = \lambda_0 / L \quad \text{with} \quad \lambda_0 = \frac{\mu_0 \sqrt{\theta_0}}{p_0} \quad (4)
\]

using a macroscopic length scale \( L \). In an asymptotic expansion, the constitutive laws (2) and (3) reduce to

\[
\mathbf{\sigma}_{\text{NSF}}^{(\text{NSF})} = -2 \mu_0 (\nabla \mathbf{v})_{\text{sym}}, \quad \mathbf{q}_{\text{NSF}}^{(\text{NSF})} = -\frac{15}{4} \mu_0 \nabla \theta, \quad (5)
\]

with which the conservation laws turn into

\[
\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{q} = 0, \quad \nabla p = \mu_0 \Delta \mathbf{v}, \quad \Delta \theta = 0, \quad (6)
\]

that is, the Stokes equations and a separate Laplace equation for the temperature. The steady, linear R13 equations can be viewed as an extension of the Stokes equations which couple the flow and temperature problem.

B. Wall boundary conditions

At an impermeable wall at rest, the linear R13 equations have to satisfy the following boundary conditions. First, we have generalized velocity slip and temperature jump conditions

\[
\sigma_{\text{slip}} = -\frac{\sqrt{\theta_0}}{\theta_0} \left( \rho_0 v_t + \frac{1}{5} q_t + \frac{1}{2} m_{\text{ent}} \right), \quad (7)
\]

\[
q_{\text{nt}} = -\frac{\sqrt{\theta_0}}{\theta_0} \left( 2 \rho_0 (\theta - \theta_0) + \frac{1}{2} \theta_0 \sigma_{\text{sm}} + \frac{5}{28} R_{\text{nt}} \right), \quad (8)
\]

which in the case of R13 are written using the variables shear stress and normal heat flux instead of velocity and temperature gradient. They also involve higher order moments.
defined below. The index \( t \) denotes the component of a tensor in the direction of the tangential flow at the wall, while \( n \) is the component normal to the wall. The components of the direction orthogonal to these two directions will be denoted by index \( s \). The coefficient \( \tilde{\chi} \) is given by \( \tilde{\chi} = \sqrt{2/\pi} \chi / (2 - \chi) \), where \( \chi \) is the accommodation coefficient of the wall model (see Ref. 18). In this paper, \( \chi = 1 \), that is, \( \tilde{\chi} = \sqrt{2/\pi} \approx 0.798 \) for full accommodation will be used.

In addition to the classical slip and jump conditions, the R13 system requires generalized slip and jump conditions for the heat flux and stress tensor

\[
R_n = -\frac{X_3}{\sqrt{\theta_0}} \left(-\theta_0 p_0 \theta_0 w_1 + \frac{11}{5} \theta_0 q_1 + \frac{1}{2} \theta_0 \sigma_{mn} \right),
\]

\[
m_{nn} = -\frac{X_3}{\sqrt{\theta_0}} \left(-\frac{2}{5} \theta_0 (\theta - \theta_0) + \frac{7}{5} \theta_0 \sigma_{mm} + \frac{1}{14} R_{nn} \right),
\]

\[
m_{nt} - m_{ss} = -\frac{X_3}{\sqrt{\theta_0}} \left(\frac{R_t - R_s}{14} + \theta_0 (\sigma_{tt} - \sigma_{ss}) \right),
\]

which involve higher order moments \( R \) and \( m \). These are given as gradients of heat flux and stress by

\[
R = -\frac{24 \mu_0}{5 \rho_0} (\nabla q)_\text{sym},
\]

\[
m = -\frac{2 \mu_0}{\rho_0} (\nabla \sigma)_{\text{sym+tracefree}}.
\]

Note that \( R \) is a symmetric and trace-free 2-tensor, according to Eq. (1c), while \( m \) is a symmetric and trace-free 3-tensor. A general definition and discussion of this tensor can be found in Ref. 13. Some components relevant to this paper are given in Sec. III C. Finally, the velocity at an impermeable wall has to satisfy

\[
v_n = 0.
\]

In total, this gives five conditions on a wall for the R13 system. For shear flows, this leads to a well-posed problem (see Ref. 18).

III. FLOW AROUND A SPHERE

We consider the slow flow of a gas around a sphere as depicted in Fig. 1. It is appropriate to use spherical coordinates with origin in the center of the sphere. The angle \( \vartheta \) runs from 0 to \( \pi \), while \( \varphi \) takes values in \([0, 2\pi]\). The flow comes from the left in the \( r-\vartheta \) plane. Due to symmetry, the flow field in the \( r-\vartheta \) plane will be the same for any value of \( \varphi \). Hence, the solutions for the fields can be assumed to be independent of \( \varphi \). In the following, all vectors and tensors will be given in spherical coordinates.

Pressure and temperature in the far field are given by the ground state. The Mach and Knudsen number for this flow are given by

\[
M_0 = \frac{u_0}{c_0}, \quad \text{Kn} = \frac{\lambda_0}{R},
\]

where \( \lambda_0 \) is the mean free path in the far field. The temperature of the sphere itself matches the temperature of the far field, that is, \( \theta_n = 0 \) in the linearized variables.

A. Classical first order solution

It is well known that for the flow around a sphere, the Stokes equation (6) admits an analytical solution (see Refs. 19 and 20). In this classical case, the Knudsen number is considered to be small. We review the solution with slip boundary conditions for later comparison.

If we assume standard first order velocity slip and temperature jump conditions at the surface of the sphere, we have

\[
v_n^{\text{slip}} = -\frac{\sqrt{\theta_0}}{\tilde{\chi} p_0} \sigma_{nt}, \quad \theta_n^{\text{jump}} = -\frac{\sqrt{\theta_0}}{2 \tilde{\chi} p_0} q_n
\]

as boundary conditions, where we used \( \theta_n = 0 \). Here, the laws of Navier–Stokes and Fourier equation (5) have been used for stress and heat flux. For \( \tilde{\chi} \to \infty \), these boundary conditions formally contain the no-slip and no-jump case, while for \( \tilde{\chi} = 0 \), the sphere is assumed adiabatic and free of shear forces. Note that also for \( \text{Kn} \to 0 \), these boundary conditions essentially describe a no-slip, no-jump wall since shear and heat flux are \( \mathcal{O} (\text{Kn}) \).

The spherical velocity field \( \mathbf{v} = (v_r, v_\vartheta, v_\varphi) \) for the Stokes equations with first order slip condition is given by

\[
v(r, \vartheta) = v_0 \left[ 1 + \frac{\tilde{\chi}}{2(3\text{Kn} + \tilde{\chi})} \frac{R^3}{r^3} - 3 \frac{2\text{Kn} + \tilde{\chi}}{2(3\text{Kn} + \tilde{\chi})} \frac{R}{r} \right] \cos \vartheta, \quad -\left[ 1 - \frac{\tilde{\chi}}{4(3\text{Kn} + \tilde{\chi})} \frac{R^3}{r^3} - 3 \frac{2\text{Kn} + \tilde{\chi}}{4(3\text{Kn} + \tilde{\chi})} \frac{R}{r} \right] \sin \vartheta, \quad 0,
\]

and for the pressure we find

\[
v(r, \vartheta) = v_0 \left[ 1 + \frac{\tilde{\chi}}{2(3\text{Kn} + \tilde{\chi})} \frac{R^3}{r^3} - 3 \frac{2\text{Kn} + \tilde{\chi}}{2(3\text{Kn} + \tilde{\chi})} \frac{R}{r} \right] \cos \vartheta, \quad -\left[ 1 - \frac{\tilde{\chi}}{4(3\text{Kn} + \tilde{\chi})} \frac{R^3}{r^3} - 3 \frac{2\text{Kn} + \tilde{\chi}}{4(3\text{Kn} + \tilde{\chi})} \frac{R}{r} \right] \sin \vartheta, \quad 0,
\]

\[
\tilde{\chi} = \sqrt{2/\pi} \chi / (2 - \chi).
\]
\[
p(r, \theta) = -3p_0M_0K_n \frac{2K_n + \bar{\chi}}{2(3K_n + \bar{\chi})} \frac{R^2}{r^2} \cos \theta
\]

as deviation from the background pressure. The spherical coordinates of the stress tensor can be computed from the velocity field and read

\[
\sigma(r, \theta) = p_0M_0K_n \begin{pmatrix}
3 \left( \frac{\bar{\chi}}{3K_n + \bar{\chi}} \right) \frac{R^4}{r^2} \cos \theta & 0 & \sigma_{\theta \theta} \\
0 & 3 \left( \frac{\bar{\chi}}{2(3K_n + \bar{\chi})} \right) \frac{R^4}{r^2} \sin \theta & 0 \\
\sigma_{\theta \theta} & 0 & \sigma_{\varphi \varphi}
\end{pmatrix},
\]

with \(\sigma_{\theta \theta} = \sigma_{\varphi \varphi} = -\frac{1}{2} \sigma_{rr}\). Note that both pressure and stress are proportional to Mach and Knudsen number. The temperature deviation and heat flux vanish in this case

\[
\theta(r, \theta) = 0, \quad q(r, \theta) = 0.
\]

The flow field is shown in Fig. 2 for two different Knudsen numbers, \(K_n=0.001\) and \(K_n=0.3\). In the case of the small Knudsen number, the slip velocity at the sphere is very small and the speed, i.e., the amplitude of the velocity, drops to zero on the surface. For high Knudsen numbers, the slip increases especially on the top and bottom of the sphere. Zero speed is only found in the front and back of the sphere. Note that the result for \(K_n=0.3\) cannot be trusted since it is beyond the validity of the Navier–Stokes and Fourier laws. We will see below how the result changes when a more accurate model is used.

### B. R13 ansatz

We follow the classical solution and propose an ansatz for the solution of the R13 equations in which the dependence on the angle \(\theta\) is made explicit by \(\sin\) and \(\cos\) functions. Vectorial and tensorial components with an odd number of indices in \(\theta\) are chosen to be proportional to \(\sin \theta\).

Scalars and components with an even number of \(\theta\) indices are proportional to \(\cos \theta\). The proportionality factor is assumed to be a general function on \(r\). Any field with an odd number of \(\varphi\) indices is assumed to be zero.

This gives for the spherical coordinates of velocity

\[
v(r, \theta) = v_0 \begin{pmatrix}
1 + a \left( \frac{r}{R} \right) \cos \theta \\
-1 - b \left( \frac{r}{R} \right) \sin \theta \\
0
\end{pmatrix},
\]

and for temperature and pressure

\[
\theta(r, \theta) = \theta_0c \left( \frac{r}{R} \right) \cos \theta, \quad p(r, \theta) = p_0d \left( \frac{r}{R} \right) \cos \theta,
\]

with unknown functions \(a(x), b(x), c(x), \) and \(d(x)\), where \(x=r/R\). For the spherical coordinates of heat flux and stress we use...
with unknown functions \( \alpha(x) \), \( \beta(x) \), \( \gamma(x) \), \( \delta(x) \), \( \kappa(x) \), and \( \sigma_{\phi\phi} = -\sigma_{r\phi} = -\sigma_{\theta\phi} \) to ensure a trace-free stress deviator. Altogether, this ansatz contains nine unknown functions. All quantities are scaled with the ground state such that the unknown functions are dimensionless. In the far field around the sphere \( r \to \infty \), all unknown function should vanish, such that equilibrium holds with pressure and temperature given by the ground state and velocity flow \( v_s = v_0(\cos \theta, -\sin \theta, 0) \) in accordance with an inflow from the left.

### C. Flow equations

Using computer algebra software, the R13 equation (1) with Eqs. (2) and (3) can easily be written in spherical coordinates evaluated on expressions (21)–(23). Most differential operators used in the equations can be found in text books for different coordinates, e.g., the divergence of a 2-tensor. In Appendix the Laplacian of a 2-tensor in spherical coordinates is given, since this expression seems not to be well known.

Because no \( \phi \) dependence is considered and the \( \theta \) dependence is explicit in the ansatz, each equation can be grouped into expressions proportional to \( \cos \theta \) or proportional to \( \sin \theta \). The only exception are the diagonal components of the equation for the stress tensor where additional expressions proportional to \( \cot \theta \) occur. These are eliminated when defining

\[
\delta(x) = -\frac{1}{2} \gamma(x),
\]

which will be used in the remainder of the paper. Interestingly, this relation means that the stress tensor in Eq. (23) shows identical normal stresses \( \sigma_{\phi\phi} = \sigma_{\theta\theta} \) in a flow around a sphere, which furthermore are related to the radial normal stress by \( \sigma_{\phi\phi} = \sigma_{\theta\theta} = -\frac{1}{2} \sigma_{rr} \). This is a feature of the stress tensor which is typically observed in purely one-dimensional processes such as shock waves, but not in shear flows. With Eq. (24), only eight unknown functions in Eqs. (21)–(23) remain.

From the R13 equations, we obtain eight ordinary differential equations with coefficients depending on \( r \) in the dimensionless form \( x := r/R \). The conservation laws give the equations

\[
a'(x) + \frac{2}{x} a(x) - \frac{2}{x} b(x) = 0,
\]

\[
d'(x) + \gamma'(x) + \frac{3}{x} \gamma(x) + \frac{2}{x} \kappa(x) = 0,
\]

\[
\kappa'(x) + \frac{3}{x} \kappa(x) - \frac{1}{2} d(x) + \frac{1}{2x} \gamma(x) = 0,
\]

which are of first differential order. The equations for stress and heat flux equations (2) and (3) contain Laplacians and relaxation terms which in spherical coordinates give rise to the differential operator

\[
\mathcal{L}(x, \lambda_1, \lambda_2) u(x) = \left( \lambda_1 + \frac{\lambda_2}{x^2} \right) u(x) - \frac{2}{x} u'(x) - u''(x)
\]

acting on a function \( u(x) \) with additional coefficients \( \lambda_1, \lambda_2 \). Using this operator as abbreviation, the equations read

\[
\frac{6Kn}{5} \left[ \mathcal{L} \left( x, \frac{5}{9Kn^2}, 4 \right) \alpha(x) - \frac{4}{x^2} \beta(x) \right] = d''(x) - \frac{5}{2} c'(x),
\]

\[
\frac{6Kn}{5} \left[ \mathcal{L} \left( x, \frac{5}{9Kn^2}, 2 \right) \beta(x) - \frac{2}{x^2} \alpha(x) \right] = \frac{1}{x} d(x) - \frac{5}{2x} c(x),
\]

\[
\frac{6Kn}{5} \left[ \mathcal{L} \left( x, \frac{5}{6Kn^2}, 2 \right) \gamma(x) - \frac{4}{9x} \kappa'(x) + \frac{68}{9x^2} \kappa(x) \right]
\]

\[
= -2M_{\phi \phi} a'(x) - \frac{4}{5} \alpha'(x),
\]

\[
\frac{16Kn}{15} \left[ \mathcal{L} \left( x, \frac{15}{16Kn^2}, 4 \right) \kappa(x) + \frac{3}{16x} \gamma'(x) + \frac{27}{8x} \gamma(x) \right]
\]

\[
= M_{\phi \phi} b'(x) - \frac{M_0}{2} a'(x) + \frac{2}{5x} a(x) + \frac{2}{5x} \beta'(x) - \frac{2}{5x} \beta(x).
\]

As expected, the differential operator \( \mathcal{L} \) acts on the components of heat flux and stress tensor \( \alpha, \beta, \gamma, \) and \( \kappa \), turning these equations into second order equations. The dimensionless parameters Mach number and Knudsen number equation (15) occur in the above equations.

The essential part of the operator \( \mathcal{L} u \) gives \( u - u'' \) and is responsible for the introduction of exponential boundary layers into the solution known as Knudsen layers (see Refs. 18 and 8). On first sight at the equations above, it seems that all components \( \alpha, \beta, \gamma, \) and \( \kappa \) will exhibit independent Knudsen layers. However, a detailed inspection shows that this is not true. The conservation laws lead to an order reduction in the following form. By using Eqs. (26) and (28), the derivative of the radial heat flux component \( \alpha \) and shear stress component \( \kappa \) can be expressed by nondifferentiated functions. Inserting this into Eqs. (30) and (33) turns these equations into first order equations, whose solution do not allow for exponential layers. However, the remaining Knudsen layers of the
angular heat flux $\beta$ and radial normal stress $\gamma$ are inherited to these quantities and all others through the coupling in the lower order equations. Note that Eq. (27) cannot be used to reduce the derivatives of $\gamma$ in Eq. (32) because additional derivatives of $d$ would occur.

After this reduction, the system contains two second order equations and six first order equations for the eight unknowns

$$
\phi(x) = [a(x), b(x), c(x), d(x), \alpha(x), \beta(x), \gamma(x), \kappa(x)].
$$

(34)

Hence, these equations require ten boundary conditions in general. This means, for instance, for a flow inside a spherical shell, five conditions on the inner and outer wall. In the case of half-infinite situations, five asymptotic conditions on the far field replace the outer wall.

D. Sphere boundary conditions

For the sphere, the normal and tangential components used in the boundary conditions, Eqs. (7)–(10), are given by the $r$ and $\theta$ components (see Fig. 1). The boundary condition (11) is identically satisfied after employing the relation (24).

In order to specify boundary conditions for the set of functions $\phi(x)$, relations (12) and (13) need to be evaluated in spherical coordinates. The relevant components of $R$ and $m$ are given by

$$
R_{r \theta} = -\frac{12 \mu_0}{5 \rho_0} (\nabla q)_{r \theta} + (\nabla q)_{\theta r},
$$

(35)

$$
R_{rr} = -\frac{24 \mu_0}{5 \rho_0} (\nabla q)_{rr},
$$

(36)

$$
m_{r r \theta} = -\frac{2 \mu_0}{\rho_0} \left( \frac{1}{3} (2(\nabla \sigma)_{r r \theta} + (\nabla \sigma)_{r \theta r}) - \frac{2}{15} (\nabla \cdot \sigma)_{\theta} \right),
$$

(37)

$$
m_{r \theta \theta} = -\frac{2 \mu_0}{\rho_0} \left( \frac{1}{3} (2(\nabla \sigma)_{r \theta \theta} + (\nabla \sigma)_{\theta r r}) - \frac{2}{15} (\nabla \cdot \sigma)_{r} \right),
$$

(38)

using, among others, the gradient of the 2-tensor $\sigma$. The spherical coordinates of $\nabla \sigma$ are provided in Appendix.

Inserting the ansatz (21)–(23) into the slip and jump conditions for velocity and temperature with $\theta_b = 0$ and $r = R$ ($x = 1$) yields

$$
- \kappa(1) = \bar{\chi}(M_0(1+b) - \frac{1}{2}\beta + \frac{8}{15} \text{Kn}(\frac{2}{3} \gamma - \kappa' + 2\kappa))_{x=1},
$$

(39)

$$
- \alpha(1) = \bar{\chi}(2c + \frac{1}{2} \gamma - \frac{6}{5} \text{Kn} \alpha')_{x=1},
$$

(40)

and the generalized slip and jump conditions read, again with $\theta_b = 0$,

$$
\frac{6}{15} \text{Kn}(3 \gamma' - 6 \gamma - 4 \kappa)_{x=1} = \bar{\chi}(\frac{2}{3} \beta + \frac{7}{3} \gamma - \frac{12}{35} \text{Kn} \alpha')_{x=1},
$$

(41)

$$
\frac{12}{5} \text{Kn}(\beta - \alpha')_{x=1} = \bar{\chi}(M_0(1+b) - \frac{11}{5} \beta + \frac{8}{15} \text{Kn}(\frac{2}{3} \gamma - \kappa' + 2\kappa))_{x=1}.
$$

(42)

The condition on the normal component of the velocity equation (14) reads

$$
1 + a(1) = 0,
$$

(43)

which gives in total five wall boundary condition on the sphere.

IV. R13 RESULTS

The general solution of the ordinary differential equations given in Eqs. (25)–(33) contains spherical Bessel functions due to the spherical operator $L$. These would be needed for the flow inside a spherical shell. For the external flow around a sphere, the solution simplifies and can be obtained with an appropriate ansatz.

A. Analytical expressions

We are looking for solutions of $\psi(x)$ of the form

$$
\psi(x) = \psi_0(x) + \psi_1(x) \exp[-\lambda_1(x-1)] + \psi_2(x) \times \exp[-\lambda_2(x-1)],
$$

(44)

where $\psi_{0,1,2}(x)$ are polynomials in $1/x$. The first part $\psi_0(x)$ represents the bulk solution, while $\psi_1(x) \exp[-\lambda_1(x-1)]$ describes the exponential boundary layers, which vanish away from the wall for $x \gg 1$. Only two layers are present according to the discussion in Sec. III C. They are inherited from the variables $\beta$ and $\gamma$ to the rest of the solution. Inspection of the equations for $\beta$ and $\gamma$ reveals that

$$
\lambda_1 = \frac{1}{\text{Kn}} \sqrt{\frac{5}{9}}, \quad \lambda_2 = \frac{1}{\text{Kn}} \sqrt{\frac{5}{6}}
$$

(45)

hold for the relaxation of the layers.

Inserting Eq. (44) into Eqs. (25)–(33), sorting the expressions which are constant or proportional to either of the exponentials, as well as collecting the powers of $1/x$, gives linear equations for the coefficients of the polynomials $\psi_{0,1,2}(x)$. The linear system is homogeneous but admits non-trivial solutions containing parameters to determine from the boundary conditions. After applying the conditions

$$
\lim_{x \to \infty} \psi(x) = 0
$$

(46)

for the far field, we find the following solutions.

It turns out that the Knudsen layers introduced by $\beta$ and $\gamma$ each influence only a separate part of the solution. The Knudsen layer of the angular heat flux $\beta$ is present in the radial heat flux $\alpha$ and both velocity components $a$ and $b$.

Their solution is given by

$$
a(x) = C_1 \frac{1}{2x} + C_2 \frac{1}{3x^3} - K_1 \left( \frac{6 \text{Kn}^3}{5x^3} + \frac{2}{\sqrt{5} x^3} \right) e^{-\frac{1}{\text{Kn}}(x-1)}.
$$

(47)
and the pressure  

\[ p = C_1 \frac{1}{4x} - C_2 \frac{1}{6x} + K_1 \left( \frac{3Kn^3}{5x^2} + \frac{Kn^2}{\sqrt{5x^2}} + \frac{Kn}{6x} \right) \times e^{-(3Kn^2(1)K_n)} \]

(48)

\[ \frac{\alpha(x)}{M_0} = C_3 \frac{Kn}{6x^2} + K_1 \left( \frac{3Kn^3}{2x^2} + \frac{\sqrt{5}Kn^2}{2x} + \frac{5Kn}{6x} \right) \times e^{-(3Kn^2(1)K_n)} \]

(49)

\[ \frac{\beta(x)}{M_0} = -C_1 \frac{Kn}{12x^3} - K_1 \left( \frac{3Kn^3}{2x^2} + \frac{\sqrt{5}Kn^2}{2x} + \frac{5Kn}{6x} \right) \times e^{-(3Kn^2(1)K_n)} \]

(50)

where wall boundary conditions at the sphere surface have not yet been employed. The Knudsen layer of the radial normal stress \( \gamma \) occurs in the shear stress \( \kappa \), the temperature \( c \), and the pressure \( d \). The solutions read

\[ \frac{c(x)}{M_0} = (C_3 + 9C_1Kn) \frac{1}{45x^2} + K_2 \left( \frac{62Kn^3}{5x^2} + \frac{2Kn}{x} \right) \times e^{-(3Kn^2(1)K_n)} \]

(51)

\[ \frac{d(x)}{M_0} = C_1 \frac{Kn}{2x} + K_2 \left( \frac{30Kn^2}{x^2} + \frac{5Kn}{x} \right) \times e^{-(3Kn^2(1)K_n)} \]

(52)

\[ \frac{\gamma(x)}{M_0} = (5C_2 + C_3Kn - 16C_1Kn^2) \times \frac{2Kn}{5x^2} + C_1 \frac{Kn}{x^2} - K_2 \left( \frac{654Kn^4}{5x^4} + \frac{54Kn^3}{x^3} \right) + \frac{4\sqrt{30Kn^2}}{x^2} + \frac{5Kn}{x} \times e^{-(3Kn^2(1)K_n)} \]

(53)

\[ \frac{\kappa(x)}{M_0} = (5C_2 + C_3Kn - 16C_1Kn^2) \frac{Kn}{5x^3} - K_2 \left( \frac{62Kn^4}{5x^4} \right) + \frac{27Kn^4}{x^3} + \frac{153Kn^2}{2x} \times e^{-(3Kn^2(1)K_n)} \]

(54)

again without wall boundary conditions. The bulk solution \( \psi_0 \) contains three integration coefficients \( C_{1,2,3} \), which are essentially computed from the classical slip and jump conditions (39) and (40) and the impermeability condition of the wall equation (43). The two Knudsen layers introduce two additional constants \( K_{1,2} \), which are specified using the generalized slip and jump conditions for heat flux and stress, Eqs. (41) and (42).

The remaining constants \( C_{1,2,3} \) and \( K_{1,2} \) depend only on the Knudsen number and the accommodation coefficient \( \chi \). They have been obtained exactly by computer algebra software, but the detailed expressions are suppressed due to their excessive length. Instead, their behavior with respect to the Knudsen number is shown in Fig. 3 for the case of full accommodation \( \chi=1 \). The limiting values for \( Kn \to 0 \) are \( C_1 = -3, C_2 = 1.5, C_3 = 0, \) and \( K_1 = -0.485, K_2 = 0 \). Note, however, that the Knudsen layer solution \( \psi_{1,2} \) is proportional to \( Kn \) and vanishes for \( Kn \to 0 \).

**B. Flow and temperature fields**

The left hand side of Fig. 4 shows the flow lines and the speed contours (amplitude of velocity) of the R13 result for the case \( Kn=0.3 \) with accommodation coefficient \( \chi=1 \). This picture can be compared to Fig. 2 where the result for the Stokes equations with first order slip conditions is shown. The flow line pattern is very similar; however, in the R13 case, the speed reaches the inflow value of unity quicker in regions above and below the sphere.

It is an interesting aspect of the solution of the R13 system that it predicts temperature variations and heat flow in the flow around a sphere. The heat flow lines together with temperature contours are shown at the right hand side of Fig. 4 for \( Kn=0.3 \). Note that the flow is considered slow and the result is based on linearized equations. Hence, quadratic energy dissipation typically responsible for temperature increase through shear is not present. The temperature variations predicted by R13 are the result of the coupling of stress and heat flux equations in Eqs. (2) and (3), which take effect for larger values of the Knudsen number. Heat fluxes in rarefied shear flows have also been reported in other works, e.g., Ref. 18.

As can be seen in Fig. 4, the temperature increases in front and decreases in the back of the sphere. Such a temperature polarization is known for rarefied or microflows around obstacles and the result of the R13 equations is in
agreement with the findings of Ref. 22 on the basis of the linearized Boltzmann equation. All quantities in the R13 result, except velocity, are proportional to the Mach number of the inflow $M_0$. The contour level values on the right hand side of Fig. 4 must be multiplied with $M_0$ to obtain the relevant temperature deviations for a specific case. The maximal temperature increase at the front of the sphere is $0.01 M_0 \theta_0$ for $Kn=0.3$.

On the top and the bottom the heat flow forms recirculation areas which indicate the Knudsen layer. Outside this layer at approximately $r=1.8$, the heat flux essentially flows from the back in big loops to the front of the sphere, in contrast to the intuition based on the temperature gradient. This is possible since the bulk solution for the temperature as given in Sec. IV A contains terms proportional to Kn not linked to the heat flux, a so-called nongradient transport effect. While the sign of the temperature polarization corresponds to the full Boltzmann solution in Ref. 22, no heat flux result for Boltzmann is given in the literature. Note that the asymptotic expansion of Ref. 21 for small Knudsen numbers predicts a higher temperature in the front, but this expansion is not valid in the case $Kn=0.3$. In the R13 result, the temperature becomes positive for very small Knudsen numbers $Kn<0.004$. Only then the hydrodynamics limit is reached in which the heat flows against the temperature gradient.

Figure 5 shows the result of the R13 equations for the case $Kn=0.9$. Again, the flow lines are shown together with speed contours, as well as heat flow lines with temperature contours. The flow lines of velocity have not changed significantly in comparison with smaller Knudsen numbers. However, the slip velocity at the top and bottom of the sphere became so large that the speed shows almost no deceleration in these regions. In fact, the slip velocity even grows beyond the inflow speed for larger values of Kn. For the heat flow lines, the recirculation areas have grown and are now almost centered outside the plot range shown on the right hand side of Fig. 5. The flow around the sphere becomes dominated by the Knudsen layer. As a result, the heat flux partly aligns with the velocity flow and fully decouples from the temperature gradient. Interestingly, the
temperature polarization has changed sign when compared to the case Kn=0.3 above. The maximal temperature increase on the sphere is now at the back and reads 0.05 $M_0\theta_b$. Such change of sign of the polarization is also reported in the Boltzmann solution for very small Knudsen numbers or when changing the accommodation coefficient (see Ref. 22 and the discussion in Ref. 21). Here, the change of sign indicates a loss of validity of the R13 equations due to the relatively large Knudsen number Kn=0.9.

In Fig. 6, we show two interesting characteristics of the flow around a sphere, the variation of the front pressure and temperature for various the Knudsen numbers. Note that also the Stokes solution (18) predicts a pressure increase in the front of the sphere and a symmetric decrease in the back which is growing with Kn. Interestingly, the R13 result gives a linear increase only for moderate Knudsen numbers and a maximal value of pressure increase is reached at Kn=0.55. For larger Knudsen numbers, the back and front pressures approach each other again. The front temperature increases with Knudsen number in a nonlinear manner showing only very small values. The temperature increase also drastically changes with the accommodation coefficient and shows a sensitivity similar to that reported in Refs. 21 and 22.

V. HYBRID STOKES-R13 SOLUTION

The general solution to the Stokes equation (6) with Eq. (5) for stress and heat flux in spherical geometry is very similar to the bulk solution $\psi_0$ of the R13 system. For the velocity and heat flux components we find

$$a(x) = C_1 \frac{1}{2x} + C_2 \frac{1}{3x^2}, \quad b(x) = C_1 \frac{1}{4x} - C_2 \frac{1}{6x^3}, \quad \frac{\alpha(x)}{M_0} = C_3 \frac{Kn}{6x^3}, \quad \frac{\beta(x)}{M_0} = -C_3 \frac{Kn}{12x^3},$$

and for temperature, pressure, and stress, the solution is

$$\frac{c(x)}{M_0} = C_3 \frac{1}{45x^2}, \quad \frac{d(x)}{M_0} = C_1 \frac{Kn}{2x^2}, \quad \gamma(x) = C_2 \frac{2Kn}{x^4} + C_1 \frac{Kn}{x^3},$$

$$\kappa(x) = C_2 \frac{Kn}{x^4},$$

where $C_{1,2,3}$ are integration constants. The condition for the far field has already been employed so that the constants need to be computed from wall boundary conditions at the sphere surface. In comparison to the R13 bulk solution in Sec. IV A, the temperature, pressure, and stress components $c, d, \gamma, \text{and } \kappa$ have less coupling to the velocity and heat flux components $a, b, \alpha,$ and $\beta$.

A. Hybrid boundary conditions

When using standard slip and jump conditions as given in Eq. (16), we obtain the classical solution as given in Sec. III A. In particular, the heat flux and temperature deviation vanishes. However, it is well known that special high order boundary conditions can extend the solution of the Navier–Stokes–Fourier equations into regimes of moderate Kn numbers (see Refs. 3 and 27). In Ref. 28, it is shown for channel flow how these higher order boundary conditions can be derived from the R13 boundary conditions by replacing the higher order moments with expressions obtained from an expansion of the R13 equations. In the present case of the flow around a sphere, we adopt this derivation for the case of the flow around a sphere. In a first step, the R13 boundary conditions without the moments $R$ and $m$ are applied to the Stokes solution leading to first order hybrid boundary conditions. The boundary conditions including higher moments are discussed in Sec. V A.

Only the first two R13 boundary conditions are relevant for the Stokes solution above and after dropping the contributions of $R$ and $m$, we write them in the form

$$v^{(\text{slip})}_0 = -\frac{\sqrt{\theta_0}}{\sqrt{P_0}} \sigma^{(\text{NSF})}_{r^2} - \frac{1}{5P_0} d^{(\text{NSF})}_{\phi},$$

$$\theta^{(\text{jump})} = -\left[\frac{\sqrt{\theta_0}}{2x^2P_0} q^{(\text{NSF})}_{xx} - \frac{\theta_0}{4P_0} \sigma^{(\text{NSF})}_{rr}\right],$$

in order to show the similarity to Eq. (16). The first order slip and jump conditions are represented by the first two terms in Eqs. (59) and (60). Using the first approximation, stress and heat flux are replaced by their Navier–Stokes–Fourier expressions. Together with the condition $v_0 = 0$, these boundary conditions then provide three conditions for the three constants $C_{1,2,3}$ of the Stokes solution above.

The final conditions read

$$\gamma(1) = M_0 (1 + b) - \frac{1}{2} \beta|_{x=1},$$

$$\alpha(1) = \chi(2c + \gamma)|_{x=1},$$

which can be compared to Eqs. (39) and (40). After inserting the general Stokes solutions (55)–(58), this gives a linear system for $C_{1,2,3}$ whose solution is again suppressed for brevity. The most significant feature is that in contrast to the first order Stokes solution, we have $C_3 \neq 0$, yielding nontrivial temperature and heat flux fields.
B. Flow and temperature fields

Figure 7 shows the result of the Stokes equation with temperature equation (6) for the flow around a sphere with hybrid boundary conditions obtained from the R13 system for Kn=0.3 and $\chi=1$. The left hand side plot depicts the velocity flow lines and speed contours. This plot can be compared to the classical first order result on the right hand side of Fig. 2 and with the full R13 result in Fig. 4. The flow field shows essentially no difference and the speed contours are very similar to the first order result. The inflow value is recovered somewhat faster toward the far field around the sphere which is the correct tendency when compared with the R13 result where the inflow value is reached much closer to the sphere.

The most relevant addition in the hybrid result is the presence of a nontrivial temperature and heat flux field. The right hand side of Fig. 7 shows the contours of temperature together with the flow lines of the heat flux for Kn=0.3. This plot can be compared to the classical first order Stokes solution, the temperature around the sphere remains constant and the heat flux vanishes. However, the hybrid Stokes solution predicts a temperature increase in the back and a decrease at the front of the sphere, which is not in agreement with the R13 result and kinetic simulations. Due to the lack of Knudsen layers and nongradient transport in the Stokes system (6), the heat flux flow field does not exhibit any recirculation regions. Instead, the flow lines are always orthogonal to the temperature contours in agreement with the law of Fourier. Interestingly, the heat flow pattern of Fig. 7 can be roughly obtained from the flow field in Fig. 4 when the Knudsen layer in the region between $r=1$ and $r=1.8$ is imagined as squeezed onto the surface. This represents the approximation obtained by neglecting the Knudsen layer inducing parts of the R13 equations. Obviously, such an approximation is only valid as long as the Knudsen layer does not influence the flow significantly as already happening for Kn=0.3.

C. Comparison of the fields

To obtain more insight into the flow around a sphere, we will show three figures with all the nonvanishing fields of the R13 solution for different Knudsen numbers and compare it to the Stokes solution based on hybrid boundary conditions as described in Sec. V A. According to the ansatz in Sec. III B, the essential part of the fields are unknown functions in radius, while the angular dependence is known. Hence, it suffices to plot the radial behavior at representative angles $\theta$.

All figures will be using the accommodation coefficient $\chi=1$. Figure 8 shows the case Kn=0.05. The R13 curves are solid, while the hybrid Stokes result is using dashed curves. The fields are radial and angular components of velocity $v_r$ and $v_\theta$, and heat flux $q_r$ and $q_\theta$, where the radial component is plotted in front of the sphere ($\theta=\pi$) and the angular component at the top ($\theta=\pi/2$). Additionally, the pressure $p$ in front and the temperature $\theta$ in the back of the sphere are displayed. In the last row, the diagonal component of the stress deviator in radial direction $\sigma_{rr}$ and its shear component $\sigma_{r\theta}$ in the $r-\theta$ plane is plotted in the back and on the top, respectively. Note that all quantities except velocity scale with the Mach number of the inflow. Temperature and pressure show the deviations from the background values.

For Kn=0.05, there is virtually no difference between the results of R13 and the hybrid Stokes solution for velocity, pressure, and stress. The velocity slip is small. Temperature and heat flux, however, show strong relative differences on a rather small absolute level. The temperature has a different sign. This shows the onset of the failure of the Stokes solution due to insufficient nonequilibrium modeling such as the lack of Knudsen layers. The Knudsen layer is clearly visible in the angular heat flux of the R13 result. A steep exponential layer between $r/R=1$ and $r/R=1.3$ with positive amplitude is superimposed to a small negative bulk solution. Only the bulk solution is visible in the Stokes result since the Stokes equations do not contain this layer behavior.
The differences between the models become stronger for larger Knudsen numbers. The fields for the case Kn=0.3 are displayed in Fig. 9 in the same way as above. The velocity slip strengthened and the Stokes solution is underestimating the absolute value of both components of the velocity, due to incorrect boundary conditions and lack of Knudsen layers. In the angular component of the heat flux, the Knudsen layer now covers a region up to $r/R=1.8$, with significant larger amplitude and can hardly be separated from the bulk. The flow becomes Knudsen layer dominated. Note that the Stokes result with hybrid boundary conditions predicts a qualitatively correct radial heat flux but an opposite sign temperature. The correct sign is brought into the R13 result by the Knudsen layers and nongradient transport effects, where temperature and heat flux are decoupled as visible in the analytical expressions. Also, the normal stress component is clearly underpredicted in the Stokes result due to the lack of Knudsen layers.

For Kn=0.9 in Fig. 10, the differences become even more pronounced. At this Knudsen number, quantitative accuracy of the R13 model cannot be expected. The R13 equations predict a nonmonotone angular velocity which is not

FIG. 8. (Color online) Details of the results of R13 and Stokes with hybrid boundary conditions for Kn=0.05. Note that the fields are shown at different $\theta$ on the sphere’s surface.
present in the hybrid Stokes result. For larger Knudsen numbers, R13 describes a slip velocity beyond the value of the inflow. Also, the heat flux components follow qualitatively different behavior. The temperature of both R13 and the hybrid Stokes solution now shows the same positive sign which is in disagreement with the Boltzmann solutions in Ref. 22 for a comparable Knudsen number. This indicates the loss of validity for the R13 equations at least in this case of the temperature field. In the figure for Kn=0.9, pressure and normal stresses also differ clearly, while the shear stress results show relatively good agreement.

D. Resulting force

In the continuum limit, the force onto a sphere in a slow flow is well known to follow from the Stokes formula

\[ F_{\text{Stokes}} = 6\pi R\mu v_0 \]

depending on the radius \( R \), the viscosity \( \mu \), and the inflow \( v_0 \). When written based on the dimensionless parameters Kn and \( M_0 \), this formula reads

\[ \text{FIG. 9. (Color online) Details of the results of R13 and Stokes with hybrid boundary conditions for Kn=0.3. Note that the fields are shown at different } \theta \text{ on the sphere’s surface.} \]
\[ F_{\text{Stokes}} = 6\pi R^2 p_0 M_0 Kn, \]

where \( p_0 \) is the far field pressure. For rarefied or microsituations, i.e., larger values of \( Kn \), this formula predicts an increasing force on the sphere which contradicts experimental observations. For \( Kn \to \infty \), the force onto the sphere should vanish.

The force can be computed from the pressure tensor \( P = p_1 + \alpha \), which has to be integrated on the surface of the sphere. The integral reads

\[ F = -2\pi R^2 \int_0^\pi e_x(\theta) \cdot (P(R, \theta) \cdot n) \sin \theta d\theta \tag{65} \]

with \( e_x(\theta) = (\cos \theta, -\sin \theta, 0)^T \) and \( n = (1, 0, 0)^T \). The vector \( P \cdot n \) represents the stress on the surface, while the scalar multiplication with \( e_x \) gives the component in the direction of the flow. Since \( P \) gives the pressure onto the fluid, the negative sign gives the force on the sphere.

For the result of the Stokes equations with first order slip conditions as described in Sec. III \( A \), we obtain.
as normalized force. For \( Kn \to 0 \) or \( \tilde{v} \to \infty \), this expression converges to unity in agreement with the no-slip solution. For larger values of the Knudsen number, the force decreases but reaches a finite value for \( Kn \to \infty \). This is still an unphysical prediction. The Stokes solution with hybrid boundary conditions yields a similar result.

Using the result of the R13 equations presented above, we find that the force indeed vanishes for large Knudsen numbers. Figure 11 shows a comparison of the different formulas over a wide range of the Knudsen number. The experimental data are taken from the empirical formula by Ref. 29 as given in Ref. 25. We would like to stress that the R13 is not expected to give quantitatively good results for values of the Knudsen number beyond unity. However, qualitative features like the vanishing force for large Knudsen numbers can be captured in contrast to classical models such as Navier–Stokes–Fourier.

E. Second order hybrid boundary conditions

Higher order boundary conditions for Navier-Stokes equations have been studied, for example, in Refs. 27 and 28. These papers considered plane channel flows and showed that higher order boundary conditions improve the Navier–Stokes solution considerably. In Ref. 28, higher order boundary conditions have been derived directly from the R13 equations. The hybrid boundary conditions above can be extended to second order by including the contributions of \( R \) and \( m \)

\[
\sigma^{(\text{slip})} = -\frac{\mu_0}{K_0} \sigma^{(2nd)} - \frac{1}{5} p_0 \theta_{(2nd)} - \frac{1}{2} \frac{\mu_0}{p_0} \mu^{(2nd)} \theta_{(2nd)},
\]

\[
\theta_{(\text{jump})} = -\frac{\mu_0}{2K_0} q^{(2nd)} - \frac{\theta_0}{4} \sigma^{(2nd)} - \frac{5}{56} R^{(2nd)} \theta_{(2nd)},
\]

and replace the moments on the right hand side by expressions obtained from a higher order asymptotic expansion in the Knudsen number of the bulk R13 equations. Such an expansion on Eqs. (2) and (3) gives for the stress tensor

\[
\sigma^{(\text{2nd order})} = -2 \frac{\mu_0}{K_0} \nabla (\nabla \cdot \mathbf{v}) - \frac{4}{5} \frac{\mu_0}{p_0} \left( \nabla q^{(\text{NSF})} \right)_{\text{sym}}
\]

\[
= \sigma^{(\text{NSF})} - \frac{4}{5} \frac{\mu_0}{p_0} \left( \nabla q^{(\text{NSF})} \right)_{\text{sym}}
\]

as second order expression and

\[
q^{(\text{2nd order})} = -\frac{15}{4} \frac{\mu_0}{p_0} \nabla \theta - \frac{3}{2} \frac{\mu_0}{p_0} \nabla \cdot \sigma^{(\text{NSF})}
\]

\[
= q^{(\text{NSF})} - \frac{3}{2} \frac{\mu_0}{p_0} \nabla \cdot \sigma^{(\text{NSF})}
\]

for the heat flux. From these we may use \( \sigma^{(\text{2nd order})} \) and \( q^{(\text{2nd order})} \) in Eqs. (59) and (60). A similar expansion on Eqs. (12) and (13) or rather, on the relevant components Eqs. (36) and (38), gives

\[
R^{(\text{2nd order})} = -\frac{24}{5} \frac{\mu_0}{p_0} \left( \nabla q^{(\text{NSF})} \right)_{rr},
\]

up to second order.

However, if these expressions are used to compute the constants \( C_{1,2,3} \) in the Stokes solution (55)–(58), the result shows no clear advantage over the hybrid first order solution in Sec. V A. While the sign of the temperature polarization turns out to be in agreement with R13, the heat flux now shows opposite direction. This indicates that the usage of higher order boundary conditions might be limited to simple interior flows where the fluid quantities do not couple as intensively as in the flow around an obstacle. Further investigation of hybrid higher order boundary conditions is left for future work.

VI. CONCLUSION

In this paper, we give analytical results for the fluid variables in a slow flow past a sphere obtained from the R13 equations. The explicit expressions contain Knudsen layers present in the flow and predict basic properties such as temperature polarization correctly for moderate Knudsen numbers. The results show that the R13 system is highly capable to describe gas flows at moderate Knudsen numbers \( Kn \lesssim 0.7 \) in an extremely efficient and straightforward way.

The paper also presents hybrid boundary conditions based on the R13 system that can be used to augment the standard Stokes equations and describe temperature polarization. However, in contrast to channel flows where higher order hybrid boundary conditions are useful, caution must be taken when applying these to more complicated flows as they do not necessarily improve the result in the general case.

With the findings of this paper, it is now straightforward to set up either numerical simulations for flows around more complicated obstacles or to enlarge the moment equations according to Ref. 15 in order to increase the accuracy further.
ACKNOWLEDGMENTS

Support through the EURYI award of the European Science Foundation (ESF) is gratefully acknowledged. The paper was written during the author’s stay as visiting professor at the Department for Applied Mathematics at the University of Washington and the author would like to thank the members of the department for their kind hospitality.

APPENDIX: SPHERICAL COORDINATES

1. Laplacian of a 2-tensor

Let \( \mathbf{V} \) be a symmetric 2-tensor in three-dimensional space. Due to symmetry, \( \mathbf{V} \) contains only six independent components. The Laplacian does not increase the tensorial degree, hence, \( \Delta \mathbf{V} \) will be a symmetric 2-tensor again. In spherical coordinates \((r, \theta, \phi)\) with \(0 \leq \theta \leq \pi\) and \(0 \leq \phi \leq 2\pi\), the six components of the Laplacian (in physical units) are given by

\[
(\Delta \mathbf{V})_{r,r} = \frac{\partial^2 \mathbf{V}_{r,r}}{r^2} + \frac{\partial \mathbf{V}_{r,r}}{r} + \frac{\partial^2 \mathbf{V}_{r,r}}{r^2 \sin^2 \theta} - \frac{4 \partial \mathbf{V}_{r,r}}{r^2 \sin \theta} + \frac{2 \mathbf{V}_{r,r}}{r^2},
\]

\[
(\Delta \mathbf{V})_{\theta,\theta} = \frac{\partial^2 \mathbf{V}_{\theta,\theta}}{r^2} + \frac{\partial \mathbf{V}_{\theta,\theta}}{r} + \frac{\partial^2 \mathbf{V}_{\theta,\theta}}{r^2 \sin^2 \theta} - \frac{4 \partial \mathbf{V}_{\theta,\theta}}{r^2 \sin \theta} + \frac{2 \mathbf{V}_{\theta,\theta}}{r^2},
\]

\[
(\Delta \mathbf{V})_{\phi,\phi} = \frac{\partial^2 \mathbf{V}_{\phi,\phi}}{r^2} + \frac{\partial \mathbf{V}_{\phi,\phi}}{r} + \frac{\partial^2 \mathbf{V}_{\phi,\phi}}{r^2 \sin^2 \theta} + \frac{2 \partial \mathbf{V}_{\phi,\phi}}{r^2 \sin \theta} + \frac{2 \mathbf{V}_{\phi,\phi}}{r^2},
\]

for the off-diagonal elements. All components of \( \mathbf{V} \) are referred to in physical units.

2. Gradient of a 2-tensor

Let \( \mathbf{V} \) be a symmetric 2-tensor as above. The gradient \( \nabla \mathbf{V} \) will be a 3-tensor symmetric in the two indices inherited from \( \mathbf{V} \), hence, containing 18 independent components. The six components of the \( r \)-derivative are given by

\[
(\nabla \mathbf{V})_{r,r} = \partial_r \mathbf{V}_{r,r}, \quad (\nabla \mathbf{V})_{r,\theta} = \partial_r \mathbf{V}_{r,\theta},
\]

\[
(\nabla \mathbf{V})_{r,\phi} = \partial_r \mathbf{V}_{r,\phi},
\]

\[
(\nabla \mathbf{V})_{\theta,\theta} = \partial_\theta \mathbf{V}_{\theta,\theta}, \quad (\nabla \mathbf{V})_{\theta,\phi} = \partial_\theta \mathbf{V}_{\theta,\phi},
\]

The six components of the \( \theta \)-derivative read

\[
(\nabla \mathbf{V})_{\theta,r} = \partial_\theta \mathbf{V}_{r,r} - \frac{2 \mathbf{V}_{r,r}}{r},
\]

\[
(\nabla \mathbf{V})_{\theta,\theta} = \frac{\partial \mathbf{V}_{r,r}}{r} + \frac{\mathbf{V}_{r,r}}{r},
\]

\[
(\nabla \mathbf{V})_{\theta,\phi} = \frac{\partial \mathbf{V}_{r,\phi}}{r} - \frac{\mathbf{V}_{r,\phi}}{r},
\]

and the \( \phi \)-derivative is given by

\[
(\nabla \mathbf{V})_{\phi,r} = \frac{\partial \mathbf{V}_{r,r}}{r \sin(\theta)} - \frac{2 \mathbf{V}_{r,r}}{r}.\]
As above, all these expressions refer to components in physical units.